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CULTURAL SCIENCES

HUMAN BEINGS AND SCIENCE–TECHNOLOGY IN THE DIGITAL AGE: PHILOSOPHICAL PERSPECTIVES ON HUMAN SUBJECTIVITY, TECHNOLOGICAL RATIONALITY, AND HUMANISTIC DEVELOPMENT

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Abstract

This paper examines the relationship between human beings and science–technology in the digital age from a philosophical perspective. It argues that digital technologies have evolved beyond mere tools to become a new environment shaping human cognition, labor, communication, and identity. While technological advances enhance creativity, knowledge access, and social development, they also generate challenges such as digital inequality, algorithmic dependence, privacy risks, and human alienation.

The paper analyzes these issues through ontological, epistemological, ethical, and social-philosophical perspectives. It proposes a human-centered approach to science and technology development, emphasizing human dignity, freedom, responsibility, and justice as fundamental values. In the context of Vietnam, the study highlights the importance of integrating digital transformation with human development, digital ethics education, and digital citizenship in building a humane digital society.

Keywords: human beings; science and technology; digital age; artificial intelligence; digital ethics.

1. Introduction

The digital age has fundamentally transformed the relationship between human beings and science–technology, raising important philosophical questions about human existence, cognition, identity, and social life. With the rapid development of artificial intelligence, big data, cloud computing, and digital networks, technology is no longer merely a tool supporting human activities; it has become a new environment in which people learn, work, communicate, and participate in society.

In this context, human beings simultaneously experience new opportunities and challenges. Digital technologies expand access to knowledge, enhance creativity, and improve social governance. At the same time, they may contribute to digital alienation, algorithmic dependence, privacy risks, and growing social inequalities. These developments require a philosophical examination of the changing relationship between human beings and technological systems.

This paper adopts an interdisciplinary philosophical approach to analyze the impacts of digital science and technology on human cognition, labor, social relations, and identity. It also discusses key ethical issues related to autonomy, responsibility, and digital justice. On this basis, the paper argues for a human-centered model of technological development in which human dignity, freedom, and comprehensive development remain the ultimate goals of digital transformation.

2. Philosophical Foundations of the Relationship Between Human Beings and Science–Technology

Throughout the history of ideas, the relationship between human beings and technical instruments has been a central philosophical concern. Unlike other living species that primarily adapt to nature, human beings transform nature through labor, knowledge, and technology. The creation of tools reflects humanity's capacity for projection: individuals conceive goals before acting, develop means to achieve those goals, and through these means transform both the world and themselves. Technology, therefore, is not external to human history but constitutes a fundamental mode through which human creativity and freedom are realized.

At the same time, modern philosophy has warned that technology may transcend its role as a mere instrument and become a system that shapes and governs human life. Heidegger argued that the essence of modern technology lies not simply in machinery itself but in the way technological thinking discloses the world. When reality is viewed primarily as a resource to be calculated, optimized, and exploited, human beings risk losing the ability to encounter things as meaningful entities in their own right[1]. This insight is particularly relevant in the digital age, where datafication may enhance efficiency while simultaneously reducing individuals to collections of data points, behavioral profiles, and predictive probabilities.

From another perspective, Marx's thought provides an important framework for understanding the relationship between technology, labor, and alienation. Technological tools can liberate labor by reducing physical burdens and expanding creative capacities. However, under specific social conditions, they may also become forces that dominate workers. Alienation emerges when labor processes, products, and rhythms are detached from human autonomy[2]. In the digital era, this phenomenon appears in new forms, as platform

[1] Martin Heidegger, "The Question Concerning Technology", trong The Question Concerning Technology and Other Essays, Harper & Row, 1977.

[2] Karl Marx, Economic and Philosophic Manuscripts of 1844, Progress Publishers, 1959.

workers may gain greater market access while being increasingly managed by opaque algorithmic systems that are difficult to understand or challenge.

Philosophy of communication and social philosophy offer additional insights. Habermas emphasized the importance of communicative action in fostering social consensus and public rationality[3]. Yet digital communication is no longer conducted solely among equal participants; it is mediated by platforms, recommendation algorithms, advertising mechanisms, and attention-based business models. As a result, the digital public sphere simultaneously expands opportunities for participation and creates risks of manipulation, polarization, and the erosion of rational dialogue.

These perspectives suggest three fundamental philosophical propositions. First, science and technology are products of human activity, yet once institutionalized as systems, they acquire the capacity to shape human life in return. Second, technology should not be evaluated solely in terms of technical efficiency but also according to its impact on human dignity, freedom, justice, and comprehensive development. Third, human beings in the digital age must be understood as reflexive subjects capable of critically assessing and regulating the technological systems they create.

3. From Instrument to Environment: The Digital Ontology of Human Existence

A defining feature of the digital age is the transformation of technology from an instrument into an environment of existence. Digital technologies are no longer merely tools used in specific situations; they increasingly constitute the infrastructure through which individuals work, learn, communicate, consume, and participate in social life. Human existence is therefore embedded within systems structured by data, algorithms, and digital interfaces.

Manuel Castells describes contemporary society as a "network society," in which power, economic activity, culture, and knowledge are organized through global information networks[4]. Within this context, an individual's opportunities are shaped not only by physical location or material resources but also by access to connectivity, information, and digital competencies. Digitalization thus creates a new social geography based on networks and flows of information.

Luciano Floridi further characterizes contemporary life through the concept of "onlife," referring to the increasingly blurred distinction between online and offline existence[5]. Personal identity is now formed through both physical interactions and digital profiles, online activities, and data traces. As a result, the self becomes simultaneously embodied, represented, and evaluated within digital environments.

These transformations raise important questions concerning authenticity, autonomy, and human agency. The proliferation of anonymous accounts, deepfakes, automated content, and algorithmically mediated interactions challenges traditional notions of authenticity. At the same time, digital environments require new ethical standards grounded in transparency, accountability, critical verification, and the protection of digital identity.

Moreover, digital technologies reshape human experiences of time and attention. Continuous flows of information, notifications, and instant communication encourage immediacy while fragmenting attention and reducing opportunities for reflection. In this context, philosophy plays a critical role in defending the space for reflective thinking, reminding us that human flourishing depends not only on rapid responses but also on the capacity for understanding, judgment, and meaningful choice.

4. Digital Science–Technology and the Transformation of Human Cognition

Digital science and technology have profoundly transformed the conditions under which human knowledge is produced, accessed, and evaluated. Whereas knowledge was traditionally organized through schools, libraries, printed texts, and professional institutions, it is now increasingly distributed through search engines, social media platforms, open databases, online learning systems, and artificial intelligence. This transformation has expanded access to information and created new opportunities for knowledge democratization. At the same time, it has generated new challenges regarding the evaluation of truth and credibility.

In digital environments, an abundance of information does not necessarily lead to deeper understanding. Individuals are often exposed to vast quantities of data without possessing the critical skills required to assess its reliability and significance. Algorithmic recommendation systems frequently prioritize engagement and attention rather than accuracy, increasing the risk of misinformation, polarization, and epistemic confusion. Consequently, a central epistemological challenge of the digital age is how to distinguish knowledge from information and information from noise.

Artificial intelligence further complicates this situation. AI systems can assist in writing, translation, data analysis, image generation, and decision support, thereby extending human cognitive capacities. However, the fluency and persuasiveness of AI-generated outputs may create an illusion of understanding, leading users to accept information uncritically. The apparent coherence of a response should not be equated with epistemic validity or truth.

For this reason, the most important cognitive competence in the digital age is not merely information retrieval but critical judgment. Individuals must be capable of questioning data sources, evaluating methods, identifying biases, and understanding the limitations of algorithmic systems. Philosophy contributes significantly to this task by cultivating reflexive thinking and critical inquiry concerning the nature, sources, and purposes of knowledge.

[3] Jürgen Habermas, The Theory of Communicative Action, Vol. 1, Beacon Press, 1984.

[4] Manuel Castells, The Rise of the Network Society, 2nd ed., Wiley-Blackwell, 2010.

[5] Luciano Floridi (ed.), The Onlife Manifesto: Being Human in a Hyperconnected Era, Springer, 2015.

Furthermore, digital cognition must be accompanied by epistemic ethics. In digital environments, individuals are not only consumers but also producers and disseminators of information. Sharing unverified content, manipulating digital materials, or generating deceptive information can have significant social consequences. Therefore, responsibility for truthfulness and informed communication has become a fundamental obligation of digital citizenship. A humane digital society requires not only technological competence but also ethical commitment to truth, accountability, and respect for others.

5. Labor, Creativity, and the Risk of Alienation in the Algorithmic Age

Digital science and technology are fundamentally reshaping the nature of work. Automation increasingly replaces routine tasks, while artificial intelligence supports data analysis, prediction, and content production. Digital platforms have also generated new forms of employment, including remote work, the gig economy, e-commerce, and platform-based services. These developments create significant opportunities for productivity, flexibility, and innovation, enabling individuals to work across borders, acquire new skills, and transform knowledge into economic and social value. At the same time, they introduce new forms of insecurity and vulnerability in the labor process.

One of the most significant challenges is algorithmic management. In many platform-based labor systems, workers are assigned tasks, evaluated, ranked, and rewarded through automated mechanisms. The criteria governing income, visibility, and job opportunities are often opaque, making managerial power less visible but no less influential. This raises important philosophical questions concerning autonomy and freedom in the workplace. When workers must continuously adapt to algorithmic requirements without understanding or contesting them, their capacity for self-determination may be diminished.

Digital alienation extends beyond industrial production and increasingly affects creative labor. Content creators often adapt their work to platform algorithms, engagement metrics, and rapidly changing audience preferences. As a result, creative expression may become oriented less toward intrinsic value and more toward visibility, popularity, and algorithmic optimization. In such circumstances, individuals risk commodifying not only their labor but also their emotions, iden--ities, and personal experiences within the attention economy.

Nevertheless, technological development should not be viewed solely through a pessimistic lens. Digital technologies can reduce physically demanding and repetitive work, expand opportunities for creativity, and facilitate lifelong learning and global collaboration. The critical issue lies not in technology itself but in how technological systems are designed, governed, and socially regulated. As Andrew Feenberg argues, technology can be democratized through public participation and the incorporation of social values into technological development[6]. Human beings should therefore be active participants in shaping technological futures rather than passive recipients of technological change.

In this context, education and labor policies must move beyond narrow skill-based training toward the cultivation of adaptive capacities. Alongside digital competencies, workers require critical thinking, lifelong learning abilities, professional ethics, and the capacity to collaborate effectively with intelligent technologies. A humane digital society should not merely ask what technology can do, but rather how technology can contribute to human flourishing and how the benefits of technological progress can be distributed more equitably.

6. Digital Ethics: Human Dignity, Freedom, and Responsibility in Technological Systems

Digital ethics has emerged as a critical concern because human actions within technological environments generate new forms of social, political, and moral consequences. Personal data can be collected, processed, and analyzed on an unprecedented scale, while automated decision-making systems increasingly influence access to education, employment, healthcare, financial services, and public resources. At the same time, misinformation, deepfakes, and synthetic content pose significant risks to individual reputation, social trust, and democratic discourse. These challenges demonstrate that digital technologies cannot be governed solely through technical expertise; they require an ethical framework centered on human dignity and fundamental rights.

Human dignity implies that individuals should never be treated merely as instruments for economic gain, data extraction, or administrative efficiency. When digital platforms collect and exploit personal data without meaningful transparency, individuals risk being reduced to resources within data-driven business models. Likewise, when artificial intelligence systems make decisions that significantly affect human lives without adequate mechanisms for explanation, accountability, or appeal, individuals become subject to forms of power that are difficult to understand and challenge. Excessive surveillance technologies may further undermine personal autonomy by encouraging self-censorship and restricting the space for independent thought and action.

Recent international standards and regulatory frameworks reflect these concerns. UNESCO's Recommendation on the Ethics of Artificial Intelligence emphasizes human rights, human dignity, transparency, fairness, and human oversight[7]. Likewise, the OECD AI Principles have been updated to promote trustworthy AI that respects human rights and democratic values[8]. The NIST AI Risk Management Framework proposes key requirements such as validity, reliability,

[6] Andrew Feenberg, Questioning Technology, Routledge, 1999.

[7] UNESCO, Recommendation on the Ethics of Artificial Intelligence, Paris, 2021.

[8] OECD, Revised Recommendation of the Council on Artificial Intelligence, OECD/LEGAL/0449, 2024.

safety, security, transparency, and accountability[9]. The European Union has adopted the AI Act based on a risk-based governance approach, under which certain AI applications posing unacceptable risks are prohibited or subject to strict regulation[10]. Collectively, these frameworks demonstrate a global shift from the question of "What can AI do?" to the question of "How should AI be used?".

However, digital ethics cannot remain a set of abstract principles. It must become a practical responsibility shared by individuals, organizations, and governments. Citizens should be equipped to protect personal data, respect the privacy of others, and use digital technologies responsibly. Technology companies should adopt principles of transparency, accountability, safety, and non-discrimination by design. Governments must establish regulatory frameworks that both encourage innovation and safeguard human rights. Educational institutions should integrate digital ethics into curricula as an essential component of digital citizenship and civic competence.

Furthermore, digital ethics requires an intergenerational perspective. Technological choices made today will shape the living conditions, opportunities, and freedoms of future generations. Data collected in the present may have long-term implications for privacy and social inclusion, while automation may fundamentally alter future labor structures. Therefore, technological development should be assessed not only in terms of short-term efficiency and economic benefit but also according to its long-term impact on human well-being, social justice, and sustainable development.

7. Digital Justice and Human Participation Rights

One of the most significant social challenges of the digital age is digital justice. Digital justice does not merely mean that everyone has access to digital devices or Internet connectivity. It also encompasses access to knowledge, digital literacy, the capacity to protect oneself in digital environments, the right to receive explanations for automated decisions, and the opportunity to participate in shaping technology-related policies. If digital transformation focuses solely on infrastructure while neglecting human capabilities, it may deepen existing social inequalities.

Digital divides are often associated with disparities in geography, income, age, gender, educational attainment, and access to public services. Urban residents may benefit more readily from digital services, whereas those in remote and disadvantaged areas often face barriers related to devices, connectivity, or digital skills. Younger generations may adapt more easily to technological change, while older adults risk exclusion from services that are increasingly moving online. Similarly, highly educated individuals may leverage artificial intelligence to enhance productivity, whereas those lacking digital competencies may become more vulnerable to displacement or dependency. Digital justice, therefore, is fundamentally a question of the equitable distribution of opportunities for development.

Digital justice is also closely linked to algorithmic bias. When training data reflect existing social inequalities, AI systems may reproduce or even amplify those inequalities. Recruitment algorithms may favor groups that have historically enjoyed advantages in the labor market. Credit-scoring systems may disadvantage individuals with limited formal financial histories. Facial recognition technologies may exhibit unequal levels of accuracy across demographic groups. Technology, therefore, is not inherently fair; fairness must be intentionally designed, continuously evaluated, and effectively governed.

From the perspective of political philosophy, digital justice requires that individuals be recognized not merely as users of technology but as active participants in its governance. Citizens should have the right to know how their data are collected and used, the right to refuse certain forms of data processing, the right to request explanations for automated decisions, and the right to seek redress when such decisions cause harm. Communities affected by technological systems should also be consulted in the formulation of technology policies. A human-centered technological order cannot be designed exclusively through top-down approaches nor be driven solely by market interests.

For this reason, digital justice is inseparable from digital democracy. Digital democracy involves more than online voting or the provision of digital public services. It refers to the expansion of meaningful citizen participation in public life through digital means while ensuring that digital spaces are not distorted by misinformation, data monopolies, or opaque political advertising. A just digital society should promote dialogue, critical engagement, and collaboration rather than merely optimizing the speed and efficiency of communication.

8. The Vietnamese Context: Digital Transformation and Human Development

In Vietnam, digital transformation has been identified as a key development strategy. The National Digital Transformation Program to 2025, with a vision toward 2030, aims to promote digital government, the digital economy, and digital society while emphasizing the creation of a safe, inclusive, and human-centered digital environment[11]. Likewise, the National Strategy on Research, Development, and Application of Artificial Intelligence to 2030 recognizes AI as a foundational technology for socio-economic development and national competitiveness[12]. These policy orientations demonstrate Vietnam's active engagement in the global technological transformation.

[9] National Institute of Standards and Technology, Artificial Intelligence Risk Management Framework (AI RMF 1.0), NIST AI 100-1, 2023.

[10] European Union, Regulation (EU) 2024/1689 laying down harmonised rules on artificial intelligence, Official Journal of the European Union, 2024.

[11] Thủ tướng Chính phủ, Quyết định số 749/QĐ-TTg ngày 03/6/2020 phê duyệt Chương trình Chuyển đổi số quốc gia đến năm 2025, định hướng đến năm 2030.

[12] Thủ tướng Chính phủ, Quyết định số 127/QĐ-TTg ngày 26/01/2021 ban hành Chiến lược quốc gia về nghiên cứu, phát triển và ứng dụng trí tuệ nhân tạo đến năm 2030.

From a developmental-philosophical perspective, however, digital transformation should not be understood merely as the digitization of procedures, data, and services. More fundamentally, it involves transforming the ways people live, learn, work, govern, and participate in society. If technological deployment does not enhance human capabilities, digital transformation may produce modern systems while leaving many citizens, small enterprises, and vulnerable groups unable to fully benefit from them. Consequently, the central objective of Vietnam's digital transformation should be the development of digital human capacities rather than infrastructure alone.

The development of digital citizens in Vietnam may be built upon three interrelated pillars. The first is universal digital competence, including the ability to use basic digital services, protect personal data, identify online fraud, and access digital knowledge. The second is critical thinking, enabling individuals to evaluate information, resist misinformation, and use artificial intelligence as a supportive tool rather than a substitute for human judgment. The third is digital ethics and culture, which require citizens not only to use technology effectively but also to engage responsibly, respect others, and contribute to a healthy digital community.

The digital age also calls for significant educational transformation. Educational institutions can no longer focus solely on transmitting fixed bodies of knowledge, as knowledge evolves rapidly and many future occupations remain undefined. Education should therefore prioritize lifelong learning, interdisciplinary thinking, technological collaboration, and ethical competence. Students should learn how to use AI tools while also understanding their limitations, including issues of copyright, plagiarism, data bias, and academic integrity. Without a strong humanistic foundation, technological proficiency alone may fail to cultivate sound judgment and responsible citizenship.

In public governance, digital technologies can improve the effectiveness, transparency, and responsiveness of public services. Data-driven approaches may enhance policy design, resource allocation, and service delivery. Nevertheless, digital governance must also address risks related to data concentration, privacy infringement, and the exclusion of individuals with limited technological capabilities. Public digital services can be considered genuinely human-centered only when they remain accessible, inclusive, explainable, and responsive to the needs of vulnerable groups.

In the cultural sphere, digital technologies present both opportunities and challenges. They enable the promotion of Vietnamese heritage, language, arts, and knowledge on a global scale. At the same time, they introduce new flows of values and cultural influences that may not always align with local traditions. Preserving cultural identity in the digital age, therefore, does not mean resisting technological change; rather, it means using technology to enrich cultural heritage while strengthening society's capacity for critical value selection. A resilient cultural identity is not one that rejects change, but one that engages constructively with change while preserving its humanistic foundations.

9. Toward a Human-Centered Approach to Science and Technology Development

The foregoing analysis suggests that the development of science and technology in the digital age should be guided by a human-centered approach. This principle is not merely a normative slogan; it entails concrete requirements for technology design, data governance, education, public policy, and social culture. A human-centered approach means that technological progress should be evaluated according to its capacity to expand human freedom, enhance capabilities, protect dignity, and promote social justice.

First, human autonomy must be safeguarded in relation to automated systems. While artificial intelligence can support decision-making, decisions with significant implications for human rights and well-being should remain subject to meaningful human oversight and accountability. Algorithms should not become the final authority in areas such as education, healthcare, justice, recruitment, and public services. Individuals should have the right to know, to receive explanations, and to challenge decisions that affect their lives.

Second, a culture of responsibility should be embedded within technological innovation. Scientists, engineers, businesses, and policymakers cannot separate ethical responsibility from technological development. A technology may be technically efficient yet socially harmful if it encourages excessive data extraction, platform dependency, discrimination, or the erosion of public trust. Therefore, ethical responsibility should be integrated throughout the entire innovation cycle, from research and design to deployment, evaluation, and governance.

Third, societies should promote digital humanistic education. Although STEM education remains essential, it is insufficient on its own. The digital age requires stronger integration between science, technology, social sciences, and the humanities. Engineers need ethical and social awareness, policymakers require technological literacy, and citizens must understand both their rights and responsibilities in digital environments. Education should not merely produce users of technology but cultivate individuals capable of understanding, critically evaluating, and responsibly applying technological tools for the common good.

Fourth, equitable access to technology must be ensured. Investments in digital infrastructure should be accompanied by support for digital skills, inclusive design, accessible communication, and protections for vulnerable groups. Technological progress cannot be considered successful if it creates new forms of exclusion. Citizens lacking digital skills should not lose access to essential services, students without adequate devices should not be deprived of educational opportunities, and workers displaced by automation should be provided with opportunities for reskilling and lifelong learning. Digital justice should therefore serve as a key criterion for evaluating digital transformation.

Finally, societies should cultivate collective reflexivity regarding technological development. Communities must continually ask critical questions: Who benefits from a particular technology? Who bears its risks? Who controls the data? Who is accountable when

harm occurs? Which values are being prioritized? Such questions help prevent societies from drifting toward technocracy, where complex social issues are reduced to problems of technical optimization. Human beings require not only efficient systems but also meaning, justice, trust, and the capacity to participate in shaping their collective future.

10. Conclusion

The relationship between human beings and science–technology in the digital age constitutes one of the central philosophical questions of the twenty-first century. Digital technologies have transformed not only the tools of everyday life but also the conditions of human existence, reshaping cognition, labor, communication, governance, and identity through data, algorithms, and interconnected networks. Consequently, technological development should be evaluated not solely in terms of efficiency and productivity, but also according to its contribution to human freedom, justice, creativity, and responsibility.

This paper has argued that science and technology possess a dual character. They can expand human capabilities, democratize access to knowledge, and promote social development; yet they may also generate digital alienation, inequality, surveillance, algorithmic dependence, and weakened social dialogue. These outcomes are not inherent to technology itself but depend on how technological systems are designed, governed, and directed. Human beings must therefore remain reflexive subjects capable of critically shaping the technologies they create.

For Vietnam, digital transformation should be closely linked to human development. A truly digital nation is not defined merely by advanced infrastructure and online services, but by citizens who possess digital competence, ethical awareness, participatory rights, and the ability to protect their dignity in technological environments. Ultimately, the stronger technology becomes, the greater the need for humanistic values. Only through the integration of scientific rationality, technological responsibility, and human dignity can a genuinely humane digital future be achieved.

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ECONOMIC SCIENCES

DETERMINANTS OF GDP GROWTH: AN ECONOMETRIC ANALYSIS OF EDUCATION, TAXATION, INVESTMENT, AND EXPENDITURE EFFECTS

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Abstract

This study employs an econometric model estimated using EViews to investigate the principal determinants of GDP growth in Albania by analyzing key economic variables in logarithmic form. The model explores the effects of education level, capital development, value-added tax (VAT), personal income tax (PIT), with a two-period lag, gross capital formation (GCF) lagged by two periods, and current expenditure (CUREXP) lagged by three periods on Albania's economic growth. Estimation results reveal that education, capital development, VAT, and current expenditure positively and significantly influence GDP growth, underscoring their critical roles in Albania's economic expansion. In particular, VAT shows the largest positive coefficient, highlighting its strong association with growth, possibly reflecting underlying economic activity or effective fiscal policy in Albania. The positive coefficient of lagged personal income tax indicates a delayed yet favorable impact on growth, thereby emphasizing the dynamic effects of taxation policies. Conversely, the negative coefficient for lagged gross capital formation suggests that past investments do not immediately translate into growth for Albania and may temporarily reduce it, potentially due to adjustment costs or inefficiencies. The negative constant term adjusts the baseline for unobserved factors affecting GDP growth. Incorporating lagged variables captures the timing and lagged nature of economic influences in Albania's context. These findings offer important insights for Albanian policymakers striving for sustainable growth, highlighting the need to enhance education, optimize taxation, and manage government expenditure to foster economic development. The data used for this analysis from 1998 - 2023 were sourced from the World Bank.

Keywords: GDP growth; econometric analysis; Albania; education; human capital; value-added tax (VAT); government expenditure; gross capital formation; fiscal policy; EViews.

1. Introduction

The Albanian economy has undergone significant transformation and development from 1998 to 2023, transitioning from a post-communist system toward a market-oriented economy. This period has been characterized by structural reforms, increased integration with global markets, and evolving fiscal and monetary policies aimed at fostering sustained economic growth. Despite progress, Albania has faced challenges, including fluctuating investment levels, fiscal constraints, and the need to improve human capital and infrastructure. Understanding the key factors driving economic growth in this context is essential for effective policymaking and long-term development.

This study aims to identify and quantify the main determinants of GDP growth in Albania between 1998 and 2023 using an econometric model incorporating key economic indicators. Specifically, the study examines how education levels, capital development, taxation policies, and government expenditure affect economic growth, accounting for delayed effects via time-lagged variables. The objective is to provide a clearer picture of the dynamic relationships among these factors and GDP growth, offering insights relevant to policymakers and stakeholders focused on Albania's economic progress.

The model analyzes the following variables: education (ED), reflecting human capital development; capital development (CD), representing physical investment; value-added tax (VAT) and personal income tax (PIT), capturing fiscal policy effects with consideration of their lagged impacts; gross capital formation (GCF), indicating investment flows with lagged influence; and current government expenditure (CUREXP), also included with a time lag to account for delayed fiscal effects. By applying logarithmic transformations and time lags, the model captures not only the immediate but also the delayed effects of these variables on Albania's GDP growth, providing a comprehensive analytical framework.

2. Literature Review

Economic growth is driven by multiple interacting factors, including human capital, investment, taxation, and government expenditure. Numerous studies have emphasized the complexity of economic growth, highlighting structural and policy influences that often unfold over time (Barro R. , 1990) (Barro, 1991; Solow, 1956).

Education is a fundamental driver of economic growth, as it enhances labor productivity, innovation, and technological adoption. Human capital theory posits that education increases the skills and efficiency of the workforce (Becker, 1964). Empirical research consistently shows positive effects of education on growth across countries and periods (Mankiw, Romer, and Weil 1992).

Taxation affects growth by influencing incentives for work, investment, and consumption. Optimal tax policy balances revenue generation with minimal distortions to economic activity (Ramsey, 1927; Diamond & Mirrlees, 1971). Value-added tax (VAT) and personal income tax (PIT) have varying growth implications depending on their levels and structures. Studies

also emphasize the importance of timing, with tax effects often materializing after delays (Auerbach, 2002; Mendoza, Razin, and Tesar, 1994).

Investment in physical capital, measured through gross capital formation, is key to expanding productive capacity and adopting new technologies. The foundational Solow growth model demonstrates the crucial role of capital accumulation in growth (Solow, 1956). Yet, investment returns can exhibit lagged effects due to adjustment costs and implementation periods (Easterly and Levine, 2001).

Government expenditure contributes to growth by financing public infrastructure and services that support private activity. However, excessive or inefficient spending may crowd out private investment (Barro, 1990). The effects of fiscal policy often appear with time lags, reflecting the delayed impact of government programs on economic output (Rodrik, 1998).

Building on this extensive literature, the current study analyzes these variables in the context of Albania from 1998 to 2023, incorporating lagged effects to capture the dynamic interactions shaping the country's economic growth.

3. Methodology

To analyze the determinants of GDP growth in Albania from 1998 to 2023, this study employs a log-linear econometric model estimated with the E-Views software. The dependent variable is the logarithm of GDP growth (LOG(GROWTH_GDP)), reflecting proportional changes in economic output over time.

The model incorporates six key explanatory variables selected based on their theoretical relevance and empirical importance in growth studies. These variables are expressed in logarithmic form to stabilize variance and interpret coefficients as elasticities:

Education (LOG(ED)): Proxy for human capital accumulation, capturing the effect of improvements in the education sector on economic productivity and growth (Becker 1964; Mankiw, Romer, and Weil 1992).

Capital Development (LOG(CD)): Represents physical capital accumulation essential to expanding production capacity, consistent with models that emphasize capital's role in growth (Solow, 1956; Easterly & Levine, 2001).

Value-Added Tax (LOG(VAT)): Reflects the influence of fiscal policy through tax revenues derived from value-added tax, a significant source of government funding affecting economic activity (Auerbach 2002; Mendoza, Razin, and Tesar 1994).

Personal Income Tax with a two-period lag (LOG(PIT(-2))): Included to capture delayed effects of personal income tax policy on economic behavior and output, recognizing fiscal impacts may unfold over time (Ramsey, 1927; Rodrik, 1998).

Gross Capital Formation with a two-period lag (LOG(GCF(-2))): Captures investment inflows impacting future productive capacity, accounting for gestation periods before investments translate into growth (Solow, 1956; Easterly & Levine, 2001).

Current Expenditure with a three-period lag (LOG(CUREXP(-3))): Measures government spending on goods and services, with a lag to account for the delayed response of economic growth to fiscal outlays (Barro, 1990; Rodrik, 1998).

The estimated equation is:

$$LOG(GROWTH_GDP) = \beta1 * LOG(ED) + \beta2 * LOG(CD) + \beta3 * LOG(VAT) + \beta4 * LOG(PIT(-2)) + \beta5 * LOG(GCF(-2)) + \beta6 * LOG(CUREXP(-3)) + C$$

where C is the constant term capturing baseline effects.

Data for all variables were collected from the World Bank's databases, ensuring consistency and coverage across the study period. The use of lagged variables reflects the dynamic nature of economic processes, where effects of taxation, investment, and public expenditure often manifest after certain time intervals (Rodrik 1998; Auerbach, 2002).

Estimation was performed with E-Views software, allowing for robust econometric analysis and the handling of time series data with appropriate lag structures. This methodological approach facilitates a nuanced understanding of short- and medium-term influences on Albania's economic growth.

3.1 Unit root test

The summary of the Group Unit Root Test for the series GROWTH_GDP, ED, CD, PIT, VAT, GCF, and CUREXP over the sample period 1998–2023 is as follows:

The test includes individual effects as exogenous variables, with automatic lag selection and Newey-West bandwidth selection using the Bartlett kernel.

Two null hypotheses are tested:

- Common unit root process (Levin, Lin & Chu test)
- Individual unit root process (Im, Pesaran and Shin W-stat, ADF Fisher Chi-square, and PP Fisher Chi-square tests)

Fig 1. Unit root test

Group unit root test: Summary
Series: GROWTH GDP, ED, CD, PIT, VAT, GCF, CUREXP
Date: 09/11/25 Time: 12:18
Sample: 1998 2023
Exogenous variables: Individual effects
Automatic selection of maximum lags
Automatic lag length selection based on SIC: 0 to 2
Newey-West automatic bandwidth selection and Bartlett kernel

Method	Statistic	Prob.**	Cross-sections	Obs
Null: Unit root (assumes common unit root process)				
Levin, Lin & Chu t*	-2.86196	0.0021	7	170
Null: Unit root (assumes individual unit root process)				
Im, Pesaran and Shin W-stat	-1.04288	0.1485	7	170
ADF - Fisher Chi-square	25.4679	0.0302	7	170
PP - Fisher Chi-square	26.3903	0.0231	7	175

** Probabilities for Fisher tests are computed using an asymptotic Chi -square distribution. All other tests assume asymptotic normality.

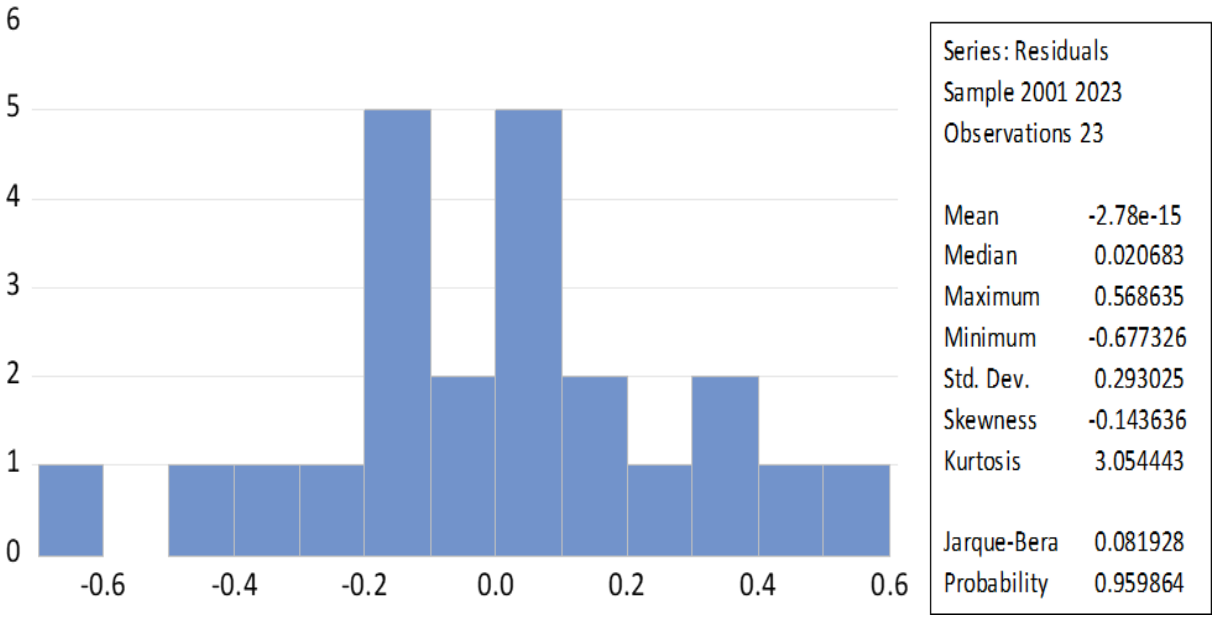
Source: Own calculation, 2025

Results:

- Levin, Lin & Chu test statistic is 2.86196 with a p-value of 0.0021, rejecting the null of a common unit root across the 7 cross-sections (series), indicating stationarity.
- Im, Pesaran and Shin W-stat is -1.04288 with a p-value of 0.1485, which does not reject the null of individual unit roots.
- ADF Fisher Chi-square statistic is 25.4679 with a p-value of 0.0302, and PP Fisher Chi-square statistic is 26.3903 with a p-value of 0.0231, both rejecting the null of unit roots for individual series.
- Overall, most tests indicate stationarity in the panel series, though the Im, Pesaran and Shin W-stat test is inconclusive at the standard significance level.

3.2 Histograma

Graph 1: Histogram of the distribution of values 1998-2023



Source: Ministry of Finance, 2025; Own Calculation, 2025

Jarque – Bera 0.081928, Probability 0.959864

Based on the data from the histogram, the skewness is equal to 0.143636, which means that the distribution is normal, and since the skewness is negative, there is a slight skew to the left. From this, we reach the important conclusion that we have a normal distribution of the values of the independent variable; therefore, the distribution of the residuals is normal.

3.3 OLS Method

Fig 2. Least Squares method

Dependent Variable: LOG(GROWTH_GDP)
Method: Least Squares
Date: 09/11/25 Time: 12:19
Sample (adjusted): 2001 2023
Included observations: 23 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
LOG(ED)	1.666880	0.806295	2.067334	0.0553
LOG(CD)	2.520398	0.666857	3.779520	0.0016
LOG(VAT)	5.036270	1.731674	2.908324	0.0103
LOG(PIT(-2))	0.383513	0.480694	0.797832	0.4366
LOG(GCF(-2))	-2.014895	0.835205	-2.412455	0.0282
LOG(CUREXP(-3))	4.910817	1.934184	2.538961	0.0219
C	-18.08936	9.407401	-1.922886	0.0725

R-squared	0.726810	Mean dependent var	1.326544
Adjusted R-squared	0.624364	S.D. dependent var	0.560625
S.E. of regression	0.343602	Akaike info criterion	0.947127
Sum squared resid	1.889001	Schwarz criterion	1.292712
Log likelihood	-3.891957	Hannan-Quinn criter.	1.034040
F-statistic	7.094553	Durbin-Watson stat	1.398369
Prob(F-statistic)	0.000801		

Source: Own calculation, 2025

The Least Squares method is a fundamental statistical technique for estimating linear regression models by minimizing the sum of squared differences between observed data and model-predicted values.

The model shows that the R-squared coefficient of determination is approximately 72.68%, i.e., close to 1, confirming that the approximation is optimal. Regarding the significance of control estimators, according to the F and t tests, we see that their values are positive, except for the VAT variable, and their probability value is less than the critical value of 5%, confirming that the model is statistically significant.

Key Variable Interpretations:

➢ LOG(ED) (Education level):

- Coefficient: 1.67, p-value ≈ 0.055 (marginally significant at 10% level)

- Interpretation: A 1% increase in education level is associated with approximately 1.67% increase in GDP growth, suggesting education positively influences economic growth.

➢ LOG(CD) (Capital development or similar):

- Coefficient: 2.52, highly significant (p=0.0016)

- Interpretation: A 1% increase in capital development leads to about 2.52% higher GDP growth, emphasizing capital accumulation's strong positive impact.

➢ LOG(VAT) (Value-added tax):

- Coefficient: 5.04, significant at 1% level (p=0.0103)

- Interpretation: A 1% rise in VAT correlates with a 5.04% increase in GDP growth, possibly indicating that increased VAT revenue may fund productive government spending, though this could require further investigation.

➢ LOG(PIT(-2)) (Personal income tax lagged two periods):

- Coefficient: 0.38, not significant (p=0.44)

- Interpretation: Personal income tax lagged two years does not have a statistically reliable impact on GDP growth in this model.

➢ LOG(GCF(-2)) (Gross capital formation lagged two periods):

- Coefficient: -2.01, significant (p=0.028) with a negative sign

- Interpretation: A 1% increase in gross capital formation two years earlier is associated with a 2% decrease in current GDP growth, which is counterintuitive and may indicate delayed or adjustment effects, or model specification issues.

➢ LOG(CUREXP(-3)) (Current expenditure lagged three periods):

- Coefficient: 4.91, significant (p=0.022)

- Interpretation: A 1% increase in government current expenditure from three years ago is associated with nearly 5% higher GDP growth, indicating that government spending has a positive lagged effect on growth.

➢ Constant (C):

- Negative but marginally significant (p=0.073), serves as baseline when explanatory variables are zero.

4. Estimation Results

The econometric model estimated using EViews provides valuable insights into the determinants of GDP growth in Albania over the period 1998–2023. The estimated equation is as follows:

LOG(GROWTH_GDP) = 1.667 * LOG(ED) + 2.520 * LOG(CD) + 5.036 * LOG(VAT) + 0.384 * LOG(PIT(-2)) - 2.015 * LOG(GCF(-2)) + 4.911 * LOG(CUREXP(-3)) - 18.089

The estimated coefficients represent elasticities, indicating the percentage change in GDP growth associated with a 1% change in each explanatory variable, holding the others constant.

Education (LOG(ED)) shows a positive coefficient of 1.667, confirming that improvements in human capital significantly foster economic growth. This result aligns with the theoretical and empirical findings that emphasize education as a critical driver of productivity and technological progress (Becker, 1964; Mankiw, Romer, and Weil, 1992).

Capital Development (LOG(CD)) presents a strong positive impact (2.520), reinforcing the importance of physical capital accumulation in expanding production capacity and supporting Albania's ongoing economic development, consistent with Solow's growth framework (Solow, 1956).

Value-Added Tax (LOG(VAT)) exhibits the largest positive coefficient (5.036), suggesting that VAT revenues or related economic activities are strongly correlated with growth. This finding highlights the significance of effective fiscal policies and taxation structures in mobilizing resources for productive investment (Auerbach, 2002; Mendoza, Razin, and Tesar, 1994).

Personal Income Tax with a two-period lag (LOG(PIT(-2))) has a positive but smaller elasticity (0.384). The lag reflects the delayed effect of PIT policy on economic decisions and growth, supporting the notion that taxation impacts may unfold over multiple periods (Ramsey, 1927; Rodrik, 1998).

Interestingly, Gross Capital Formation lagged by two periods (LOG(GCF(-2))) has a negative coefficient (-2.015). This suggests that investments may not immediately translate into growth and could temporarily restrain economic output, possibly due to adjustment costs or inefficient allocation (Easterly and Levine, 2001). It indicates the need for careful timing and efficiency in investment policies.

Current Expenditure lagged by three periods (LOG(CUREXP(-3))) shows a significant positive effect (4.911), underscoring the stimulative role of government spending after a lag. This reflects that fiscal expenditure can enhance growth when effectively targeted and timed (Barro, 1990; Rodrik, 1998).

The negative constant term (-18.089) accounts for baseline factors not explicitly included in the model.

These results collectively demonstrate that Albania's GDP growth is shaped by education, capital accumulation, fiscal policy, and government expenditure, with time lags capturing delayed responses. The findings provide a quantitative basis for policy interventions focused on enhancing human capital, optimizing taxation, managing investment efficiency, and appropriately timing government spending to support sustainable growth.

5. Discussion

The findings of this study offer important insights into the dynamics of economic growth in Albania from 1998 to 2023, emphasizing the multifaceted roles of education, taxation, investment, and government expenditure.

The strong positive impact of education reinforces the critical importance of investing in human capital to enhance labor productivity and innovation capacity. Consistent with Becker (1964) and Mankiw, Romer, and Weil (1992), policies that improve access to quality education and skills development are vital for sustaining long-term economic growth in Albania. Enhancing educational outcomes can thus be a cornerstone of national development strategies.

The positive and substantial coefficient for capital development highlights the continued relevance of physical investment in infrastructure and productive capacity, confirming Solow's (1956) assertion that capital accumulation drives growth. However, the negative coefficient observed for lagged gross capital formation suggests that investments must be both timely and efficient. This may reflect adjustment costs, delays in project implementation, or suboptimal allocation that temporarily constrains growth (Easterly & Levine, 2001). Policymakers should therefore focus on improving the effectiveness and management of investment projects, ensuring they translate more quickly and positively into economic output.

The significant positive effects of VAT and lagged personal income tax underscore the dual role of fiscal policy as both a source of government revenue and an instrument influencing economic behavior. The strong influence of VAT indicates that effective tax collection and a well-structured tax system contribute substantially to growth by providing resources for public goods and investment (Auerbach, 2002; Mendoza, Razin, and Tesar 1994). The lagged positive impact of personal income tax reinforces the notion that the effects of taxation manifest over time, highlighting the need for medium- to long-term fiscal planning (Ramsey, 1927; Rodrik, 1998).

Government current expenditure also demonstrates a robust lagged positive effect on growth, confirming that fiscal spending, when targeted and timed appropriately, can stimulate economic activity and enhance overall productivity (Barro, 1990; Rodrik, 1998). However, the presence of lags suggests that expenditure programs require prudent design and execution, with consideration of timing to maximize their effectiveness.

Together, these results imply a coordinated policy approach for Albania that integrates education reforms, efficient capital investment, and prudent fiscal management. Recognizing the delayed effects of taxation and expenditure is essential for policymakers to implement forward-looking strategies that balance short-term constraints with long-term growth objectives.

Although the model offers valuable insights, some limitations should be acknowledged. The negative sign on lagged investment calls for further investigation into the quality and sectors of capital formation. Additionally, the model does not account for external shocks, institutional quality, or other structural factors that may influence growth. Future research could expand the model to include these dimensions.

6. Conclusion

This study examined the key determinants of GDP growth in Albania over the period 1998 to 2023 using an econometric model estimated with the E-Views software. By analyzing education, capital development, value-added tax, personal income tax, gross capital formation, and current government expenditure—with time lags to capture delayed effects—the research sheds light on the dynamic factors shaping Albania's economic trajectory.

The results highlight education and capital development as significant positive drivers of growth, emphasizing the importance of human capital accumulation and physical investment. Taxation, particularly value-added tax and lagged personal income tax, also plays a crucial role, providing necessary fiscal resources while influencing economic behavior with delayed effects. Current government expenditure contributes positively to growth after a lag, illustrating the stimulative potential of well-targeted fiscal policies. Conversely, the negative coefficient on lagged gross capital formation suggests that investment efficiency and timing require careful attention to avoid short-term growth constraints.

These findings carry important policy implications for Albania. Strengthening education systems, improving the efficiency of capital investment, optimizing tax structures, and managing government expenditure with attention to timing can collectively foster sustainable economic growth. Recognizing and planning for the temporal dynamics of these influences is critical for effective economic policymaking.

While this study advances understanding of growth determinants in Albania, further research incorporating broader structural factors and external influences could provide a more comprehensive perspective. Nonetheless, the model offers a valuable framework for policymakers aiming to design strategies that enhance the country's growth potential in the coming years.

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MATHEMATICAL SCIENCES

RESEARCH OF BICONVEX FUNCTIONS

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ИССЛЕДОВАНИЕ БИВЫПУКЛЫХ ФУНКЦИЙ

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Abstract

This paper studies biconvex functions, which are pointwise upper bounds for some nonempty family of 2-affine, 2-linear, biaffine and bilinear functions, respectively. It is shown that biconvex functions are closely related to convex functions defined on the tensor product of spaces. This paper investigates the bisubdifferentiability of biconvex functions and studies several of their properties. The paper also studies the properties of biconjugate functions. The paper also studies the bisubdifferential of the maximum function. Using the second conjugate function on the tensor product of spaces, criteria for bisubdifferentiability of functions are obtained. The characterization of biconvex functions is considered.

Аннотация

В работе изучаются бивыпуклые функции, являющиеся поточечной верхней границей некоторого непустого семейства 2-аффинных, 2-линейных, биаффинных и билинейных функций соответственно. Показано, что бивыпуклые функции тесно связаны с выпуклыми функциями, определенными на тензорном произведении пространств.

В работе исследуется бисубдифференцируемость бивыпуклых функций и изучается ряд их свойств. В работе также изучаются свойства бисопряженных функций. В статье изучается бисубдифференциал функции максимума. Используя вторую сопряженную функцию на тензорном произведении пространств, получены критерии бисубдифференцируемости функ-ций. Рассматривается характеризация бивыпуклых функций.

Keywords: biconvex function, tensor product, maximum function.
Ключевые слова: бивыпуклая функция, тензорное произведение, функция максимума.

1. Основные понятия и обозначения

Бивыпуклые функции особенно важны при изучении субдифференцируемости второго порядка невыпуклых функций.

В конце 1970-х годов область выпуклого анализа значительно расширилась, и многие из его методов получили существенное развитие и стали классическими (см. [1-7]).

Некоторые свойства бисублинейных и бивыпуклых функций изучались автором в работах [8-14]. Изучению бивыпуклых функций и бивыпуклых множеств посвящена монография [11]. Многие статьи и препринты автора посвящены этой тематике (см. [11]), также ссылку на литературу в [11] по этой теме). Но класс, бивыпуклых функций очень широк. Поэтому еще имеются неисследованные задачи.

В изучении субдифференцируемости выпуклых функций существенную роль играет теорема об отделимости выпуклых множеств, а в исследовании бисубдифференцируемости бивыпуклых функций существенную роль играет понятие тензорного произведения пространств [15], [16]. Отделимость выпуклых множеств рассмотривается в [1-7, 17].

Данная работа является продолжением работы [18,19] автора.

В исследовании необходимого условия второго порядка для негладких задач оптимизации существенно используется бисубдифференцируемость бисублинейных функций.

Пусть X и Y действительные банаховы пространства, $R_+ = [0,+\infty)$, $R_{+\infty} = R \bigcup \{+\infty\}$, $R_{-\infty} = R \bigcup \{-\infty\}$, $\overline{R} = R \bigcup \{\pm\infty\}$. Обозначим через $X \otimes Y$ тензорные произведения пространств X и Y (см. [15], [16]). Функцию $f : X \times Y \to R_{+\infty}$ назовем бивыпуклой, если функции $f(\cdot, y) : X \to R_{+\infty}$, $f(x,\cdot) : Y \to R_{+\infty}$ выпуклые при $x \in X$ и $y \in Y$. Такое определение бивыпуклой функции охватывает достаточно многие функции. Например, если $f : X \times Y \to R_{+\infty}$ выпуклая функция, то она также является бивыпуклой. Если $g_1 : X \to R_{+\infty}$ и $g_2 : Y \to R_{+\infty}$ выпуклые неотрицательные функции, то $g_1(x)g_2(y)$ также является бивыпуклой функцией на $X \times Y$, сужение на пространстве $X \times Y$ выпуклых функций определенных на $X \otimes Y$ и ряд других классов функции являются бивыпуклыми.

Функция $q: X \to R_{+\infty}$ называется положительно однородной, если $q(\lambda x) = \lambda q(x)$ для всех $x \in X$ и $\lambda > 0$. Функция $q: X \to R_{+\infty}$ называется сублинейной, если она положительно однородна, выпукла и $q(0) = 0$.

Пусть $g: X \times Y \to R_{+\infty}$ и $(x, y) \in X \times Y$. Если функции $x \to g(x, y)$, $y \to g(x, y)$ положительно однородные и $g(0, y) = g(x, 0) = 0$ при $(x, y) \in X \times Y$, то функцию g назовем биположительно однородной.

Если функции $x \to g(x, y)$, $y \to g(x, y)$ выпуклые положительно однородные и $g(0, y) = g(x, 0) = 0$ при $(x, y) \in X \times Y$, то функцию g назовем бисублинейной.

Если $-g$ бисублинейная функция, то функцию $g: X \times Y \to R_{-\infty}$ назовем бисуперлинейной.

Если функции $x \to g(x, y)$, $y \to g(x, y)$ выпуклые, а функции $x \to g(x, 0)$, $y \to g(0, y)$ сублинейные и $g(0,0) = 0$, то функцию g назовем 2-сублинейной.

Функцию $f: X \times Y \to R_{-\infty}$ назовем бивогнутой, если функции $f(\cdot, y): X \to R_{-\infty}$, $f(x, \cdot): Y \to R_{-\infty}$ вогнутые при $x \in X$ и $y \in Y$.

Отметим, что бивыпуклые, бивогнутые, бисублинейные и бисуперлинейные функции из $X \times Y$ в $\overline{R}$ определяются аналогично.

Множество бивыпуклых функций из $X \times Y$ в R является конусом, который обозначим через K. Легко проверяется, что $K - K$ является линейным пространством.

Обозначим через $B(X \times Y, R)$ векторное пространство билинейных непрерывных функций из $X \times Y$ в R. Пусть $x^* \in B(X \times Y, R)$, $z^* \in (X \times Y)^*$, $\alpha \in R$. Известно, что $z^*(x, y) = z_1^*(x) + z_2^*(y)$ при $x \in X$ и $y \in Y$, где $(z_1^*, z_2^*) \in X^* \times Y^*$. Функцию $b(x, y) = x^*(x, y) + z_1^*(x) + z_2^*(y) + \alpha$ назовем 2-аффинной, функцию $b(x, y) = x^*(x, y) + z_1^*(x) + z_2^*(y)$ назовем 2-линейной, где $(x, y) \in X \times Y$.

Если $x^* \in B(X \times Y, R)$, $c \in R$, то функцию $b(x, y) = x^*(x, y) + c$ назовем биаффинной.

Множество всех биаффинных функций из $X \times Y$ в R обозначим $Ba(X \times Y)$.

Если множество $\operatorname{dom} f = \{(x, y) \in X \times Y : f(x, y) < +\infty\}$ непусто, то функция $f: X \times Y \to R_{+\infty}$ называется собственной.

Особенно отметим четыре важных подкласса бивыпуклых функций.

1) Совокупность собственных функций $f: X \times Y \to R_{+\infty}$, каждая из которых является поточечной верхней гранью некоторого непустого семейства 2-аффинных функций обозначим через $A(X \times Y)$. Из этого определения немедленно вытекает, что все функции $f \in A(X \times Y)$ бивыпуклы и полунепрерывны снизу.

2) Совокупность собственных функций $f: X \times Y \to R_{+\infty}$, каждая из которых является поточечной верхней гранью некоторого семейства биаффинных функций обозначим через $AB(X \times Y)$.

3) Совокупность собственных функций $f: X \times Y \to R_{+\infty}$, каждая из которых является поточечной верхней гранью некоторого семейства билинейных непрерывных функций обозначим через $SB(X \times Y)$.

4) Совокупность собственных функций $f: X \times Y \to R_{+\infty}$, каждая из которых является поточечной верхней гранью некоторого семейства 2-линейных непрерывных функций обозначим через $SA(X \times Y)$.

Считаем, что $X \otimes Y$ снабжено проективной топологией (см. [15, с.88]. Если $f_1(x \otimes y) = f_2(x \otimes y)$ при $(x, y) \in X \times Y$, то отождествив функции $f_1: X \otimes Y \to R_{+\infty}$ и $f_2: X \otimes Y \to R_{+\infty}$, получим взаимно однозначное соответствие между классом всех функций, определенных на $X \times Y$ и $X \otimes Y$ соответственно, значения которых совпадают. Поэтому если возможно, изучение свойств функций, определенных на $X \times Y$ можно привести к изучению свойств функций, определенных на $X \otimes Y$.

Отметим также четыре важных подкласса выпуклых функций.

а) Множество всех собственных полунепрерывных снизу выпуклых функций из $X \otimes Y \times X \times Y$ в $R_{+\infty}$ обозначим через $\Gamma_0(X \otimes Y \times X \times Y)$.

б) Множество всех собственных полунепрерывных снизу выпуклых функций из $X \otimes Y$ в $R_{+\infty}$ обозначим через $\Gamma_0(X \otimes Y)$.

с) Множество всех собственных полунепрерывных снизу сублинейных функций из $X \otimes Y$ в $R_{+\infty}$ обозначим через $S(X \otimes Y)$.

d) Множество всех собственных полунепрерывных снизу сублинейных функций из $X \otimes Y \times X \times Y$ в $R_{+\infty}$ обозначим через $S(X \otimes Y \times X \times Y)$.

Легко проверяется, что между классами $A(X \times Y)$ и $\Gamma_0(X \otimes Y \times X \times Y)$, между классами $AB(X \times Y)$ и $\Gamma_0(X \otimes Y)$, между классами $SB(X \times Y)$ и $S(X \otimes Y)$, между классами $SA(X \times Y)$ и $\Gamma_0(X \otimes Y \times X \times Y)$ существуют тесные связи. Если $g \in \Gamma_0(X \otimes Y \times X \times Y)$ и $\varphi(x, y) = g(x \otimes y, x, y)$ при

$(x,y)\in X\times Y$, то $\varphi\in A(X\times Y)$. Если $g\in\Gamma_0(X\otimes Y)$ и $\varphi(x,y)=g(x\otimes y)$ при $(x,y)\in X\times Y$, то $\varphi\in AB(X\times Y)$. Если $g\in S(X\otimes Y)$ и $\varphi(x,y)=g(x\otimes y)$ при $(x,y)\in X\times Y$, то $\varphi\in SB(X\times Y)$. Если $g\in S(X\otimes Y\times X\times Y)$ и $\varphi(x,y)=g(x\otimes y,x,y)$ при $(x,y)\in X\times Y$, то $\varphi\in SA(X\times Y)$.

Если $f\in A(X\times Y)$, то функцию f назовем (A)-выпуклой.

Если $\varphi\in AB(X\times Y)$ или $\varphi(x,y)=g(x\otimes y)$ при $(x,y)\in X\times Y$, где $g\in\Gamma_0(X\otimes Y)$, то функцию φ назовем канонической бивыпуклой. Если $\varphi\in SB(X\times Y)$ или $\varphi(x,y)=g(x\otimes y)$ при $(x,y)\in X\times Y$, где $g\in S(X\otimes Y)$, то функцию φ назовем канонической бисублинейной. Отметим, что $SB(X\times Y)\subset AB(X\times Y)\subset A(X\times Y)$. Каноническая бивыпуклая функция является бивыпуклой и полунепрерывной снизу, каноническая бисублинейная функция является бисублинейной и полунепрерывной снизу.

Функцию $z^*(x,y)=z_1^*(x)+z_2^*(y)+\alpha$ назовем аффинной, где $(x,y)\in X\times Y$, $(z_1^*,z_2^*)\in X^*\times Y^*$. Совокупность собственных функций $f:X\times Y\to R_{+\infty}$, каждая из которых является поточечной верхней гранью некоторого семейства аффинных функций обозначим через $\Gamma_0(X\times Y)$. Из этого определения немедленно следует, что все функции $f\in\Gamma_0(X\times Y)$ выпуклы полунеп-рерывны снизу. Класс выпуклых функций из $X\times Y$ в $R_{+\infty}$ хорошо изучен. В работе в основ-ном изучается бисубдифференцируемость бивыпуклых функций, которая «существенно» отличается от субдифференцируемости выпуклых функций.

Пусть X и Y линейные пространства. Также можно рассмотреть следующий подкласс бивыпуклых функций. Сужение на пространстве $X\times Y$ выпуклых функций определенных на пространстве $X\otimes Y\times(X\times Y)$ и сужение на пространстве $X\times Y$ выпуклых функций определенных на $X\otimes Y$ бивыпуклая функция, сужение на пространстве $X\times Y$ сублинейных функций определенных на пространстве $X\otimes Y$ бисублинейная функция. Если $f(\nu,x,y)$ выпуклая функция на $X\otimes Y\times(X\times Y)$, то $\varphi(x,y)=f(x\otimes y,x,y)$ при $(x,y)\in X\times Y$ бивыпуклая функция на $X\times Y$. Если $f_1(\nu)$ выпуклая функция на $X\otimes Y$, то $\varphi_1(x,y)=f_1(x\otimes y)$ при $(x,y)\in X\times Y$ бивыпуклая функция на $X\times Y$. Если $g(\nu)$ сублинейная функция на $X\otimes Y$, то $q(x,y)=g(x\otimes y)$ при $(x,y)\in X\times Y$ бисублинейная функция на пространстве $X\times Y$.

Обозначим через $\widetilde{B}(X\times Y,R)$ векторное пространство билинейных функций из $X\times Y$ в R. Пусть $z_1^*:X\to R$, $z_2^*:Y\to R$ линейные функции, $x^*\in\widetilde{B}(X\times Y,R)$, $\alpha\in R$. Функцию $b(x,y)=x^*(x,y)+z_1^*(x)+z_2^*(y)+\alpha$ назовем алгебраически 2-аффинной в $X\times Y$. Функция $f:X\times Y\to R_{+\infty}$, которая, является поточечной верхней гранью некоторого семейства алгебраически 2-аффинных функций является бивыпуклой функцией.

Работа состоит из четырех пунктов. В п.2 изучаются свойства бивыпуклых функций. В п.2 исследуется бисубдифференцируемость бивыпуклых функций и изучается ряд их свойств. В п.2 изучаются также свойства бисопряженных функций. В п.2 получены эквивалентные определения бисубдифференциала бивыпуклых функций в терминах бисопряженных функций.

В п.3 определяется Bb-выпуклое множество в терминах 2(B)-отделимости и расматри-вается ряд Bb-выпуклых множеств, которые определяются с помощью бивыпуклых функций.

В п.3 изучается ряд свойств бивыпуклых множеств. Доказана теорема об 2(B)-отделимости двух бивыпуклых множеств. Рассматривается характеризация бивыпуклых функций, когда $2-\operatorname{ep} f$ Bb-выпуклое множество.

В п.4 дается определение 2-сублинейных функций и изучается ряд их свойств.

2. Исследование свойств бивыпуклых функций

В п.2 исследована бисубдифференцируемость бивыпуклых функций и изучен ряд их свойств.

Пусть X и Y действительные банаховы пространства. Множество всех непрерывных билинейных функций из $X\times Y$ в R обозначим через $B(X\times Y,R)$.

Функцию $f:X\times Y\to R_{+\infty}$ назовем (строго, сильно) бивыпуклой (или 2-выпуклой), если функции $x\to f(x,y)$, $y\to f(x,y)$ (строго, сильно) выпуклые при $x\in X$ и $y\in Y$.

Если $g_1:X\to R_{+\infty}$ и $g_2:Y\to R_{+\infty}$ (строго, сильно) выпуклые неотрицательные (положительные) функции, то $g_1(x)g_2(y)$ также является (строго, сильно) бивыпуклой функцией на $X\times Y$.

Функцию $f:X\times Y\to R_{-\infty}$ назовем (строго, сильно) бивогнутой, если функции $f(\cdot,y):X\to R_{+\infty}$, $f(x,\cdot):Y\to R_{+\infty}$ (строго, сильно) вогнутые при $x\in X$ и $y\in Y$.

В определениях сильно бивыпуклых и сильно бивогнутых функций предполагется, что X и Y гильбертово пространство.

Если $f:X\times Y\to R_{+\infty}$ бивыпуклая функция, то $(-f):X\times Y\to R_{-\infty}$ -бивогнутая функция.

Множество $C \subset X \times Y$ называется бивыпуклой, если для любого $(\overline{x},\overline{y}) \in C$ множества $C_{\overline{x}} = \{y : (\overline{x}, y) \in C\}$, $C_{\overline{y}} = \{x : (x,\overline{y}) \in C\}$ выпуклые или из $(x_1, y), (x_2, y) \in C$, $(x, y_1), (x, y_2) \in C$ и $\alpha \in [0,1]$ следуют, что $(\alpha x_1 + (1-\alpha)x_2, y) \in C$, $(x, \alpha y_1 + (1-\alpha)y_2) \in C$.

Если $C \subset X \times Y$ бивыпуклое множество и $(\overline{x},\overline{y}) \in X \times Y$, то $C - (\overline{x},\overline{y})$ бивыпуклое множество.

Пересечение бивыпуклых множеств бивыпукло. Пустое множество считаем бивыпуклым.

Если $f_i : X \times Y \to R_{+\infty}$ бивыпуклые функции при $i = 1,\dots,k$ и $\alpha_i \in R$, то $C = \{(x,y) \in X \times Y : f_i(x,y) \le \alpha_i, i = 1,\dots,k\}$ бивыпуклое множество. Поэтому, если $x_i^* \in B(X \times Y, R)$, $v_i^* \in X^*$, $y_i^* \in Y^*$, $\alpha_i \in R$, где $i = 1,\dots,k$, то $D = \{(x,y) \in X \times Y : x_i^*(x,y) + v_i^*(x) + y_i^*(y) \le \alpha_i, i = 1,\dots,k\}$ бивыпуклое множество. Множество $D = \{(x,y) \in X \times Y : x_i^*(x,y) + v_i^*(x) + y_i^*(y) \le \alpha_i, i = 1,\dots,k\}$ назовем полэдральным бивыпуклым множеством в $X \times Y$.

Если $f : X \times Y \to R_{+\infty}$ бивыпуклая функция, то $\operatorname{dom} f = \{(x,y) \in X \times Y : f(x,y) < +\infty\}$ бивыпуклое множество.

Пусть X и Y действительные линейные пространства. Обозначим (см. [15,16]) через $X\Theta Y$ пространство формальных линейных комбинаций (с действительными коэффициентами) элементов $X \times Y$. Употребляя запись $x\Theta y$ вместо $1(x,y)$ для элементов естественного базиса в $X\Theta Y$, рассмотрим множество $M \subset X\Theta Y$ элементов любого из следующих видов:

$$(x_1 + x_2)\Theta y - x_1\Theta y - x_2\Theta y; \quad x\Theta(y_1 + y_2) - x\Theta y_1 - x\Theta y_2; \quad \lambda x\Theta\mu y - \lambda\mu x\Theta y;$$

взятых по всем $x_1, x_2, x \in X$, $y_1, y_2, y \in Y$, $\lambda, \mu \in R$. Введем обозначения $X \otimes Y$ для фактор-пространства $X\Theta Y / \operatorname{Lin} M$, где $\operatorname{Lin} M$ линейная оболочка множества M. Если $x \in X$, $y \in Y$, то класс эквивалентности, содержащий $x\Theta y$, обозначим $x \otimes y$, т.е. обозначим через $x \otimes y$ класс смежности $x\Theta y + \operatorname{Lin} M$. Поэтому

$$x \otimes y = x\Theta y + \operatorname{Lin} M = x\Theta y + \operatorname{Lin}\{(x_1 + x_2)\Theta\overline{y} - x_1\Theta\overline{y} - x_2\Theta\overline{y},\ \overline{x}\Theta(y_1 + y_2) - \overline{x}\Theta y_1 - \overline{x}\Theta y_2,$$
$$\lambda_1 z_1 \Theta \lambda_2 z_2 - \lambda_1\lambda_2 z_1 \Theta z_2 : \overline{x}, x_1, x_2, z_1 \in X,\ \overline{y}, y_1, y_2, z_2 \in Y,\ \lambda_1, \lambda_2 \in R\}.$$

Если X, Y – банахово пространство, то считаем, что пространство $X \otimes Y$ снабжено топологией, порожденной относительно нормы

$$\|v\|_p = \inf\{\sum_{i=1}^{m} \|x^i\| \|y^i\| : \ v = \sum_{i=1}^{m} x^i \otimes y^i,\ x^i \in X,\ y^i \in Y,\ m \in N\},$$

т.е. $X \otimes Y$ снабжено проективной топологией (см.[16]). Далее, будем отождествлять (см. [16]) $(X \otimes Y)^*$ и $B(X \times Y, R)$. Из предложения 3.21[16] следует, что $\|x \otimes y\| = \|x\| \|y\|$ при $(x,y) \in X \times Y$.

Если $f \in A(X \times Y)$, то существует множество индексов T такое, что

$$f(x,y) = \sup_{t \in T}(x_t^*(x,y) + v_t^*(x) + y_t^*(y) + \alpha_t),$$

при $(x,y) \in X \times Y$, где $x_t^* \in B(X \times Y, R)$, $v_t^* \in X^*$, $y_t^* \in Y^*$, $\alpha_t \in R$. Ясно, что

$$g(v,x,y) = \sup_{t \in T}(\sum_{i=1}^{k} x_t^*(x^i, y^i) + v_t^*(x) + y_t^*(y) + \alpha_t),$$

принадлежит $\Gamma_0(X \otimes Y, X, Y)$, где $v = \sum_{i=1}^{k} x^i \otimes y^i$, $k \in N$. Обозначив $S = \{(x_t^*, v_t^*, y_t^*, \alpha_t) : t \in T\}$ имеем, что $f(x,y) = \sup_{(x^*,v^*,y^*,\alpha) \in S}\{x^*(x,y) + v^*(x) + y^*(y) + \alpha\}$, где $S \subset B(X \times Y, R) \times X^* \times Y^* \times R$.

Отметим, что если $g(x,y) = \sup_{t \in T_1}(x_t^*(x,y) + \alpha_t) + \sup_{\tau \in T_2}(v_\tau^*(x) + y_\tau^*(y) + \beta_\tau)$ собственная функция, то $g \in A(X \times Y)$, где $x_t^* \in B(X \times Y, R)$, $v_\tau^* \in X^*$, $y_\tau^* \in Y^*$, $\alpha_\tau \in R$, $\beta_\tau \in R$.

Обратно, если $f \in \Gamma_0(X \otimes Y \times X \times Y)$, то из предложения 1.3.1[17, с.24] следует, что функция $f(v,x,y)$ является поточечной верхней гранью некоторого непустого семейства аффинных функций вида $b(v,x,y) = \sum_i x^*(x^i, y^i) + v^*(x) + y^*(y) + \alpha$, где $x^* \in B(X \times Y, R)$, $\alpha \in R$, $(v^*, y^*) \in X^* \times Y^*$ и $v = \sum_i x^i \otimes y^i$. Поэтому существует множество индексов T такое, что $f(v,x,y) = \sup_{t \in T} b_t(v,x,y) = \sup_{t \in T}(\sum_{i=1}^{k} x_t^*(x^i, y^i) + v_t^*(x) + y_t^*(y) + \alpha_t)$, $k \in N$. Отсюда следует, что $\varphi(x,y) = f(x \otimes y, x, y) = \sup_{t \in T}(x_t^*(x,y) + v_t^*(x) + y_t^*(y) + \alpha_t)$ принадлежит в $A(X \times Y)$, т.е. $\varphi \in A(X \times Y)$. Получим, что

$$A(X \times Y) = \{\varphi(x,y) = f(x \otimes y, x, y) \text{ при } (x,y) \in X \times Y : f \in \Gamma_0(X \otimes Y \times X \times Y)\}.$$

Пусть $\varphi(x,y)=\sup\limits_{t\in T}(x_t^*(x,y)+v_t^*(x)+y_t^*(y)+c_t)$ принадлежит в $A(X\times Y)$, $x^*\in X^*$, $y^*\in Y^*$ и $x^*(\bar{x})=1$, $y^*(\bar{y})=1$. Тогда $g((x,u,\alpha),(y,\upsilon,\beta))=\sup\limits_{t\in T}(x_t^*(x,y)+v_t^*(x)y^*(\upsilon)+x^*(u)y_t^*(y)+c_t\alpha\beta)$ бисублинейная функция в пространстве $(X\times X\times R)\times(Y\times Y\times R)$ и $g((x,\bar{x},l),(y,\bar{y},l))=f(x,y)$ при $(x,y)\in X\times Y$.

Если $g_1:X\to R_{+\infty}$ и $g_2:Y\to R_{+\infty}$ выпуклые неотрицательные непрерывные функции, то $g_1(x)g_2(y)$ в общем случае не принадлежит в $A(X\times Y)$.

Если $f:X\to R_{+\infty}$ собственная полунепрерывная снизу выпуклая функция, то из предложения 1.3.1[17, с.24] следует, что $f(x)=\sup\limits_{(p,\alpha)\in S_f}\{\langle p,x\rangle+\alpha\}$, где

$S_f=\{(p,\alpha)\in X^*\times R:\langle p,x\rangle+\alpha\le f(x)$ при $x\in X\}$. Отметим, что $\widetilde{f}(x,t)=\sup\limits_{(p,\alpha)\in S_f}\{\langle p,x\rangle+\alpha t\}$ -сублинейная функция в пространстве $X\times R$ и $\widetilde{f}(x,l)=f(x)$ при $x\in X$.

Пусть $S_1\subset X^*\times R$ и $S_2\subset Y^*\times R$ четные множества, $f(x)=\sup\limits_{(p,\alpha)\in S_1}\{\langle p,x\rangle+\alpha\}$ при $x\in X$ и $g(y)=\sup\limits_{(q,\beta)\in S_2}\{\langle q,y\rangle+\beta\}$ при $y\in Y$ собственные выпуклые функции. По условию имеем, что

$$f(x)g(y)=\sup_{(p,\alpha)\in S_1}\{\langle p,x\rangle+\alpha\}\sup_{(q,\beta)\in S_2}\{\langle q,y\rangle+\beta\}=\sup_{(p,\alpha)\in S_1}|\langle p,x\rangle+\alpha|\sup_{(q,\beta)\in S_2}|\langle q,y\rangle+\beta|=$$

$$=\sup_{(p,\alpha)\in S_1}\sup_{(q,\beta)\in S_2}|\langle p,x\rangle+\alpha||\langle q,y\rangle+\beta|=\sup_{(p,\alpha)\in S_1,\,(q,\beta)\in S_2}(\langle p,x\rangle+\alpha)(\langle q,y\rangle+\beta)$$

при $(x,y)\in X\times Y$.

Если $h_1:X\to R_{+\infty}$ и $h_2:Y\to R_{+\infty}$ собственные полунепрерывные снизу четные сублинейные функции, то $h_1(x)h_2(y)$ собственная полунепрерывная снизу четная бисублинейная функция и принадлежит в $SB(X\times Y)$ (см. [11, с.93]) и

$$h_1(x)h_2(y)=\sup_{p\in\partial h_1,\,q\in\partial h_2}\langle p,x\rangle\langle q,y\rangle$$

при $(x,y)\in X\times Y$, где $\partial h_1=\{p\in X^*:h_1(x)\ge\langle p,x\rangle$ при $x\in X\}$, $\partial h_2=\{q\in Y^*:h_2(y)\ge\langle q,y\rangle$ при $y\in Y\}$.

Отметим, что если $f(\nu,x,y)$ является поточечной супремума некоторого непустого семейства аффинных функций вида $b(\nu,x,y)$, то из предложения 1.3.1[17, с.24] следует, что $f(\nu,x,y)$ выпуклая полунепрерывная снизу функция в $X\otimes Y\times(X\times Y)$.

Лемма 2.1 доказана в [18, см. лемма 2.1].

Лемма 2.1. Если $f:X\otimes Y\times(X\times Y)\to R_{+\infty}$ выпуклая функция, то $\varphi(x,y)=f(x\otimes y,x,y)$ бивыпуклая функция на $X\times Y$.

Отсюда также следует, что если $f\in\Gamma_0(X\otimes Y\times X\times Y)$, то $\varphi(x,y)=f(x\otimes y,x,y)$ при $(x,y)\in X\times Y$ бивыпуклая функция на $X\times Y$.

Если $f\in\Gamma_0(X\otimes Y\times X\times Y)$, то положив $\varphi(x\otimes y,x,y)=f(x\otimes y,x,y)$ при $(x\otimes y,x,y)\in X\otimes Y\times X\times Y$, $\varphi(\nu,x,y)=+\infty$ при $(\nu,x,y)\ne(x\otimes y,x,y)$, $(\nu,x,y)\in X\otimes Y\times X\times Y$, рассмотрим выпуклую оболочку функции φ. Обозначим

$$X\otimes Y\oplus X\oplus Y=\{(\nu,x,y)\in X\otimes Y\times X\times Y:\alpha_i\ge0,\ \sum_{i=1}^k\alpha_i=1,\ (\nu,x,y)=\sum_{i=1}^k\alpha_i(x^i\otimes y^i,x^i,y^i),\ (x^i,y^i)\in X\times Y,\ k\in N\}.$$

Так как $ep(\operatorname{conv}\varphi)=\operatorname{conv}ep\,\varphi$, то из теоремы 1.6.1[1, с.162] следует, что

$$\operatorname{conv}\varphi(\nu,x,y)=\breve{f}(\nu,x,y)=\inf\{\sum_{i=1}^k\alpha_i\varphi(x^i\otimes y^i,x^i,y^i):\alpha_i\ge0,\ \sum_{i=1}^k\alpha_i=1,(\nu,x,y)=$$

$$=\sum_{i=1}^k\alpha_i(x^i\otimes y^i,x^i,y^i),\ (x^i,y^i)\in X\times Y,\ k\in N\}=$$

$$=\inf\{\sum_{i=1}^k\alpha_i f(x^i\otimes y^i,x^i,y^i):\alpha_i\ge0,\ \sum_{i=1}^k\alpha_i=1,(\nu,x,y)=\sum_{i=1}^k\alpha_i(x^i\otimes y^i,x^i,y^i),\ (x^i,y^i)\in X\times Y,\ k\in N\}\ge$$

$$\ge\inf\{f(\sum_{i=1}^k\alpha_i(x^i\otimes y^i,x^i,y^i):\alpha_i\ge0,\ \sum_{i=1}^k\alpha_i=1,\ \ (\nu,x,y)=$$

$$=\sum_{i=1}^k\alpha_i(x^i\otimes y^i,x^i,y^i),\ (x^i,y^i)\in X\times Y,\ k\in N\}=f(\nu,x,y)$$

при $(\nu,x,y)\in X\otimes Y\oplus X\oplus Y$. Отсюда имеем, что $\breve{f}(x\otimes y,x,y)\geq f(x\otimes y,x,y)$ при $(x\otimes y,x,y)\in X\otimes Y\times X\times Y$. Так как $\breve{f}(x\otimes y,x,y)\leq f(x\otimes y,x,y)$ при $(x\otimes y,x,y)\in X\otimes Y\times X\times Y$, то $\breve{f}(x\otimes y,x,y)=f(x\otimes y,x,y)$ при $(x\otimes y,x,y)\in X\otimes Y\times X\times Y$.

Так как $\inf\varnothing=+\infty$, то считаем, что $f(\nu,x,y)=+\infty$ при $(\nu,x,y)\notin X\otimes Y\oplus X\oplus Y$.

Если $f:X\times Y\to R_{+\infty}$, то положим

$$2-\operatorname{ep} f=\{((x,\alpha),(y,\beta))\in(X\times R)\times(Y\times R):\ f(x,y)\leq\alpha\beta\}.$$

Лемма 2.2 доказана в [18, см. лемма 2.2].

Лемма 2.2. Функция $f:X\times Y\to R_{+\infty}$ бивыпуклая функция тогда и только тогда, когда множество $2-\operatorname{ep} f$ бивыпуклое множество.

Из леммы 2 следует, что бивыпуклую функцию можно определять следующим образом: функция $f:X\times Y\to R_{+\infty}$ называется бивыпуклой, если $2-\operatorname{ep} f$ бивыпуклое множество.

Отметим, что если $((\overline{x},\alpha),(\overline{y},\beta))\in 2-\operatorname{ep} f$, то $((\overline{x},\overline{y}),\alpha\beta)\in\operatorname{ep} f=\{((x,y),\lambda)\in(X\times Y)\times R: f(x,y)\leq\lambda\}$. Поэтому можно положить $2-E(f)=\{(x,y,\alpha,\beta)\in(X\times Y)\times(R\times R):\ f(x,y)\leq\alpha\beta\}$.

Лемма 2.3. Функция $f:X\times Y\to R_{+\infty}$ бивыпуклая функция тогда и только тогда, когда для любого $(\overline{x},\overline{y})\in\operatorname{dom} f$ множества $\operatorname{ep} f(\overline{y})=\{((x,\overline{y}),\lambda)\in(X\times Y)\times R: f(x,\overline{y})\leq\lambda\}$ и $\operatorname{ep} f(\overline{x})=\{((\overline{x},y),\lambda)\in(X\times Y)\times R: f(\overline{x},y)\leq\lambda\}$ выпуклые.

Доказательство. Пусть f бивыпуклая функция и $(\overline{x},\overline{y})\in\operatorname{dom} f$. Покажем, что $\operatorname{ep} f(\overline{y})$ и $\operatorname{ep} f(\overline{x})$ выпуклые множества. Если $((x_1,\overline{y}),\lambda_1)),((x_2,\overline{y}),\lambda_2))\in\operatorname{ep} f$ и $\nu\in[0,1]$, то покажем, что $\nu((x_1,\overline{y}),\lambda_1)+(1-\nu)((x_2,\overline{y}),\lambda_2)\in\operatorname{ep} f$. Так как $f(x_1,\overline{y})\leq\lambda_1$, $f(x_2,\overline{y})\leq\lambda_2$, то имеем, что

$$f(\nu x_1+(1-\nu)x_2,\overline{y})\leq\nu f(x_1,\overline{y})+(1-\nu)f(x_2,\overline{y})\leq\nu\lambda_1+(1-\nu)\lambda_2,$$

т.е. $f(\nu x_1+(1-\nu)x_2,\overline{y})\leq\nu\lambda_1+(1-\nu)\lambda_2$. Отсюда следует, что

$$((\nu x_1+(1-\nu)x_2,\overline{y}),\nu\lambda_1+(1-\nu)\lambda_2)\in\operatorname{ep} f.$$

Если $((\overline{x},y_1),\beta_1)),((\overline{x},y_2),\beta_2))\in\operatorname{ep} f$ и $\mu\in[0,1]$, то аналогично проверяется, что $((\overline{x},\mu y_1+(1-\mu)y_2),\mu\beta_1+(1-\mu)\beta_2)\in\operatorname{ep} f$. Получим, что $\operatorname{ep} f(\overline{y})$ и $\operatorname{ep} f(\overline{x})$ выпуклые множества.

Пусть $\operatorname{ep} f(\overline{y})$ и $\operatorname{ep} f(\overline{x})$ выпуклые множества. Покажем, что f бивыпуклая функция, т.е. покажем, что $x\to f(x,y)$ и $y\to f(x,y)$ выпуклые функции.

Пусть $(x_1,y),(x_2,y)\in\operatorname{dom} f$ и $\nu\in[0,1]$. Так как $((x_1,y),f(x_1,y))$, $((x_2,y),f(x_2,y))$ принадлежит в $\operatorname{ep} f$, то имеем, что $((\nu x_1+(1-\nu)x_2,y),\nu f(x_1,y)+(1-\nu)f(x_2,y))\in\operatorname{ep} f$ при $\nu\in[0,1]$. Поэтому

$$f(\nu x_1+(1-\nu)x_2,y)\leq\nu f(x_1,y)+(1-\nu)f(x_2,y)$$

при $(x_1,y),(x_2,y)\in\operatorname{dom} f$ и $\nu\in[0,1]$. Случай $(x_1,y)\notin\operatorname{dom} f$ или $(x_2,y)\notin\operatorname{dom} f$, а также $(x_1,y)\notin\operatorname{dom} f$ и $(x_2,y)\notin\operatorname{dom} f$ очевиден. Получим, что $x\to f(x,y)$ выпуклая функция. Аналогично проверяется, что $y\to f(x,y)$ выпуклая функция. Лемма доказана.

Пусть $f:X\times Y\to R_{+\infty}$ и $(\overline{x},\overline{y})\in\operatorname{dom} f$. Положим

$$\begin{aligned}
&D f(\overline{x},\overline{y})=\{(x^*,x_1^*,y_1^*)\in B(X\times Y,R)\times X^*\times Y^*: f(x,y)-f(\overline{x},\overline{y})\geq\\
&\geq x^*(x,y)+x_1^*(x)+y_1^*(y)-x^*(\overline{x},\overline{y})-x_1^*(\overline{x})-y_1^*(\overline{y})\ \text{при}\ (x,y)\in X\times Y\},\\
&D_2 f(\overline{x},\overline{y})=\{x^*\in B(X\times Y,R):\ \exists(x_1^*,y_1^*)\in X^*\times Y^*,\ \text{что}\ f(x,y)-f(\overline{x},\overline{y})\geq\\
&\geq x^*(x,y)+x_1^*(x)+y_1^*(y)-x^*(\overline{x},\overline{y})-x_1^*(\overline{x})-y_1^*(\overline{y})\ \text{при}\ (x,y)\in X\times Y\},
\end{aligned}$$

$$\partial^2 f(\overline{x},\overline{y})=\{x^*\in B(X\times Y,R): f(\overline{x}+x,\overline{y}+y)-2f(\overline{x},\overline{y})+f(\overline{x}-x,\overline{y}-y)\geq 2x^*(x,y)\ \text{при}\ (x,y)\in X\times Y\}.$$

Лемма 2.4 доказана в [18, см. лемма 2.3].

Лемма 2.4. Если $(x^*,x_1^*,y_1^*)\in Df(\overline{x},\overline{y})$, то $x^*(\cdot,\overline{y})+x_1^*\in\partial_x f(\overline{x},\overline{y})$ и $x^*(\overline{x},\cdot)+y_1^*\in\partial_y f(\overline{x},\overline{y})$.

Если $(x^*,x_1^*,y_1^*)\in Df(\overline{x},\overline{y})$, то $f(x,0)-f(\overline{x},\overline{y})\geq -x^*(\overline{x},\overline{y})+x_1^*(x)-x_1^*(\overline{x})-y_1^*(\overline{y})$ при $x\in X$ и $f(0,y)-f(\overline{x},\overline{y})\geq -x^*(\overline{x},\overline{y})-x_1^*(\overline{x})+y_1^*(y)-y_1^*(\overline{y})$ при $y\in Y$. Поэтому

$f(x,0)-x_1^*(x)\geq -x^*(\overline{x},\overline{y})+f(\overline{x},\overline{y})-x_1^*(\overline{x})-y_1^*(\overline{y})$ и $f(0,y)-y_1^*(y)\geq -x^*(\overline{x},\overline{y})+f(\overline{x},\overline{y})-x_1^*(\overline{x})-y_1^*(\overline{y})$ при $x\in X$, $y\in Y$.

Лемма 2.5 доказана в [18, см. лемма 2.4].

Лемма 2.5. Если $(x^*,x_1^*,y_1^*)\in Df(\overline{x},\overline{y})$, то

$$f(\overline{x}+x,\overline{y}+y)-2f(\overline{x},\overline{y})+f(\overline{x}-x,\overline{y}-y)\geq 2x^*(x,y)$$

при $(x,y)\in X\times Y$, т.е. $x^*\in\partial^2 f(\bar{x},\bar{y})$.

Если $f_1: X\times Y\to R_{+\infty}$, $f_2: X\times Y\to R_{+\infty}$, $Df_1(\bar{x},\bar{y})\neq\varnothing$ и $Df_2(\bar{x},\bar{y})\neq\varnothing$, то легко проверяется, что $Df_1(\bar{x},\bar{y})+Df_2(\bar{x},\bar{y})\subset D(f_1+f_2)(\bar{x},\bar{y})$.

Отметим, что $D_2 f(\bar{x},\bar{y})\subset\partial^2 f(\bar{x},\bar{y})$.

Если $\lambda>0,\mu>0$ и $x^*\in\partial^2 f(\bar{x},\bar{y})$, то

$$f(\bar{x}+\lambda x,\bar{y}+\mu y)-2f(\bar{x},\bar{y})+f(\bar{x}-\lambda x,\bar{y}-\mu y)\geq 2x^*(\lambda x,\mu y)$$

при $(x,y)\in X\times Y$. Поэтому если $x^*\in\partial^2 f(\bar{x},\bar{y})$, то

$$f^-((\bar{x},\bar{y});(x,y))=\inf_{\lambda>0,\mu>0}\tfrac{1}{\lambda\mu}(f(\bar{x}+\lambda x,\bar{y}+\mu y)-2f(\bar{x},\bar{y})+f(\bar{x}-\lambda x,\bar{y}-\mu y))\geq 2x^*(x,y)$$

при $(x,y)\in X\times Y$.

Если $f: X\times Y\to R_{+\infty}$ и $(\bar{x},\bar{y})\in\operatorname{dom} f$, то обозначим

$$\varphi(x,y)=f(\bar{x}+x,\bar{y}+y)-2f(\bar{x},\bar{y})+f(\bar{x}-x,\bar{y}-y)$$

при $(x,y)\in X\times Y$.

Если $f: X\times Y\to R_{+\infty}$ бивыпуклая функция, $(\bar{x},\bar{y})\in\operatorname{dom} f$, $(x_1,y),(x_2,y)\in X\times Y$, $\alpha\in[0,1]$, то

$$\varphi(\alpha x_1+(1-\alpha)x_2,y)=f(\bar{x}+\alpha x_1+(1-\alpha)x_2,\bar{y}+y)-2f(\bar{x},\bar{y})+f(\bar{x}-\alpha x_1-(1-\alpha)x_2,\bar{y}-y)=$$
$$=f(\alpha(\bar{x}+x_1)+(1-\alpha)(\bar{x}+x_2),\bar{y}+y)-2f(\bar{x},\bar{y})+f(\alpha(\bar{x}-x_1)+(1-\alpha)(\bar{x}-x_2),\bar{y}-y)\leq$$
$$\leq\alpha f(\bar{x}+x_1,\bar{y}+y)+(1-\alpha)f(\bar{x}+x_2,\bar{y}+y)-2\alpha f(\bar{x},\bar{y})-2(1-\alpha)f(\bar{x},\bar{y})+\alpha f(\bar{x}-x_1,\bar{y}-y)+$$
$$+(1-\alpha)f(\bar{x}-x_2,\bar{y}-y)=\alpha\varphi(x_1,y)+(1-\alpha)\varphi(x_2,y),$$

т.е. функция $x\to\varphi(x,y)$ выпукла. Аналогично проверяется, что $y\to\varphi(x,y)$ выпукла. Получим, что если $f: X\times Y\to R_{+\infty}$ бивыпуклая функция и $(\bar{x},\bar{y})\in\operatorname{dom} f$, то $\varphi(x,y)$ бивыпуклая функция.

Если $f: X\times Y\to R_{+\infty}$ бивыпуклая функция и $(\bar{x},\bar{y})\in\operatorname{dom} f$, то положим

$$f^{(2)+}((\bar{x},\bar{y});(x,y))=\limsup_{\lambda\downarrow 0,\mu\downarrow 0}\frac{\varphi(\lambda x,\mu y)}{\lambda\mu}$$

при $(x,y)\in X\times Y$, т.е. положим

$$f^{(2)+}((\bar{x},\bar{y});(x,y))=\limsup_{\lambda\downarrow 0,\mu\downarrow 0}\tfrac{1}{\lambda\mu}(f(\bar{x}+\lambda x,\bar{y}+\mu y)-2f(\bar{x},\bar{y})+f(\bar{x}-\lambda x,\bar{y}-\mu y))$$

при $(x,y)\in X\times Y$.

Лемма 2.6 доказана в [18, см. лемма 2.5].

Лемма 2.6. Если $f: X\times Y\to R$ бивыпуклая функция, то $x\to f^{(2)+}((\bar{x},\bar{y});(x,y))$, $y\to f^{(2)+}((\bar{x},\bar{y});(x,y))$ выпуклые положительно однородные функции и $f^{(2)+}((\bar{x},\bar{y});(-x,-y))=f^{(2)+}((\bar{x},\bar{y});(x,y))$ при $(x,y)\in X\times Y$.

Положим $\bar{\partial}^2 f(\bar{x},\bar{y})=\{x^*\in B(X\times Y,R): f^{(2)+}((\bar{x},\bar{y});(x,y))\geq 2x^*(x,y)$ при $(x,y)\in X\times Y\}$.

Легко проверяется, что $\partial^2 f(\bar{x},\bar{y})\subset\bar{\partial}^2 f(\bar{x},\bar{y})$.

Обозначим через B_X и B_Y единичные замкнутые шары в X и Y соответственно, т.е. $B_X=\{x\in X:\|x\|\leq 1\}$, $B_Y=\{y\in Y:\|y\|\leq 1\}$.

Бивыпуклая функция $f: X\times Y\to R$ называется раздельно билипшицевой с постоянной L в окрестности точки $(\bar{x},\bar{y})$, если существует число $\delta>0$ такое, что

$$|f(\bar{x}+x,\bar{y}+y)-2f(\bar{x},\bar{y})+f(\bar{x}-x,\bar{y}-y)|\leq L\|x\|\cdot\|y\|$$

при $x\in\delta B_X$, $y\in\delta B_Y$.

Если $f: X\times Y\to R$ раздельно билипшицевая функция с постоянной L в окрестности точки $(\bar{x},\bar{y})$, то $f(\bar{x}+x,\bar{y})-2f(\bar{x},\bar{y})+f(\bar{x}-x,\bar{y})=0$, $f(\bar{x},\bar{y}+y)-2f(\bar{x},\bar{y})+f(\bar{x},\bar{y}-y)=0$ при $x\in X$ и $y\in Y$, т.е. свойство раздельно билипшицевости функции в окрестности точки $(\bar{x},\bar{y})$ выполняется при жестких условиях.

Примеры. 1) Если $C\subset X$ и $D\subset Y$ выпуклые множества, $\bar{x}\in C$ и

$$d_C(x)=\inf\{\|c-x\|:c\in C\}\quad\text{и}\quad d_D(y)=\inf\{\|z-y\|:z\in D\},$$

то $(x,y)\to d_C(x)d_D(y)$ бивыпуклая функция и

$$|d_C(\bar{x}+x)d_D(\bar{y}+y)+d_C(\bar{x}-x)d_D(\bar{y}-y)|\leq 2\|x\|\cdot\|y\|$$

при $(x,y)\in X\times Y$.

2) Если $g_1 : X \to R_{+\infty}$ и $g_2 : Y \to R_{+\infty}$ выпуклые неотрицательные полунепрерывные снизу функции, то $g_1(x)g_2(y)$ также бивыпуклая функция в $X \times Y$ и если $\bar{x} \in \operatorname{int} \operatorname{dom} g_1$ и $\bar{y} \in \operatorname{int} \operatorname{dom} g_2$ и $g_1(\bar{x}) = 0$, то существуют $\delta > 0$ и $L > 0$ такие, что

$$\left|g_1(\bar{x}+x)g_2(\bar{y}+y) + g_1(\bar{x}-x)g_2(\bar{y}-y)\right| \le L\|x\| \cdot \|y\|$$

при $(x,y) \in X \times Y$, $\|x\| \le \delta$, $\|y\| \le \delta$.

3) Если $C \subset X \otimes Y$ выпуклое множество и $\bar{x} \otimes \bar{y} \in C$, то $(x,y) \to d_C(x \otimes y)$ бивыпуклая функция в $X \times Y$ и

$$\left|d_C(\bar{x} \otimes \bar{y} + x \otimes y) + d_C(\bar{x} \otimes \bar{y} - x \otimes y)\right| \le 2\|x\| \cdot \|y\|$$

при $(x,y) \in X \times Y$.

4) Если существует множество индексов T такое, что

$$f(x,y) = \sup_{t \in T}(x_t^*(x,y) - x_t^*(x-\bar{x},\bar{y}) - x_t^*(\bar{x},y-\bar{y}) + \alpha_t),$$

при $(x,y) \in X \times Y$, где $x_t^* \in B(X \times Y, R)$, $\alpha_t \in R$ и $\left|x_t^*(x,y)\right| \le L\|x\|\|y\|$ при $t \in T$, то

$$f(\bar{x}+x,\bar{y}+y) - 2f(\bar{x},\bar{y}) + f(\bar{x}-x,\bar{y}-y) = \sup_{t \in T}(x_t^*(\bar{x}+x,\bar{y}+y) - x_t^*(x,\bar{y}) - x_t^*(\bar{x},y) + \alpha_t) -$$

$$- 2\sup_{t \in T}(x_t^*(\bar{x},\bar{y}) + \alpha_t) + \sup_{t \in T}(x_t^*(\bar{x}-x,\bar{y}-y) + x_t^*(x,\bar{y}) + x_t^*(\bar{x},y) + \alpha_t) =$$

$$= \sup_{t \in T}(x_t^*(\bar{x},\bar{y}) + x_t^*(x,y) + \alpha_t) - 2\sup_{t \in T}(x_t^*(\bar{x},\bar{y}) + \alpha_t) + \sup_{t \in T}(x_t^*(\bar{x},\bar{y}) + x_t^*(x,y) + \alpha_t) \le$$

$$\le 2\sup_{t \in T} x_t^*(x,y) \le 2L\|x\|\|y\|$$

при $(x,y) \in X \times Y$. Аналогично имеем, что

$$f(\bar{x}+x,\bar{y}+y) - 2f(\bar{x},\bar{y}) + f(\bar{x}-x,\bar{y}-y) = \sup_{t \in T}(x_t^*(\bar{x},\bar{y}) + x_t^*(x,y) + \alpha_t) -$$

$$- 2\sup_{t \in T}(x_t^*(\bar{x},\bar{y}) + \alpha_t) + \sup_{t \in T}(x_t^*(\bar{x},\bar{y}) + x_t^*(x,y) + \alpha_t) \ge 2\inf_{t \in T} x_t^*(x,y) \ge -2L\|x\|\|y\|$$

при $(x,y) \in X \times Y$. Поэтому $\left|f(\bar{x}+x,\bar{y}+y) - 2f(\bar{x},\bar{y}) + f(\bar{x}-x,\bar{y}-y)\right| \le 2L\|x\|\|y\|$ при $(x,y) \in X \times Y$.

Если бивыпуклая функция $f : X \times Y \to R$ раздельно билипшицевая с постоянной L в окрестности точки $(\bar{x},\bar{y})$, то $\left|f^{(2)+}((\bar{x},\bar{y});(x,y))\right| \le L\|x\| \cdot \|y\|$ при $(x,y) \in X \times Y$. Тогда, если $x^* \in \bar{\partial}^2 f(\bar{x},\bar{y})$, то $f^{(2)+}((\bar{x},\bar{y});(x,y)) \ge 2x^*(x,y)$ и $f^{(2)+}(\bar{x},\bar{y};-x,y) \ge -2x^*(x,y)$ при $(x,y) \in X \times Y$. Поэтому $\left|2x^*(x,y)\right| \le L\|x\| \cdot \|y\|$ при $x^* \in \bar{\partial}^2 f(\bar{x},\bar{y})$ и $(x,y) \in X \times Y$, т.е. множество $\bar{\partial}^2 f(\bar{x},\bar{y})$ ограничено.

Если $f : X \times Y \to R$ бивыпуклая функция, $0 < \lambda_1 \le \lambda_2$ и $0 < \mu_1 \le \mu_2$, то

$$f(\bar{x}+\lambda_1 x,\bar{y}+\mu_1 y) - 2f(\bar{x},\bar{y}) + f(\bar{x}-\lambda_1 x,\bar{y}-\mu_1 y) =$$

$$= f(\tfrac{\lambda_1}{\lambda_2}(\bar{x}+\lambda_2 x) + (1-\tfrac{\lambda_1}{\lambda_2})\bar{x}, \tfrac{\mu_1}{\mu_2}(\bar{y}+\mu_2 y) + (1-\tfrac{\mu_1}{\mu_2})\bar{y}) - 2f(\bar{x},\bar{y}) +$$

$$+ f(\tfrac{\lambda_1}{\lambda_2}(\bar{x}-\lambda_2 x) + (1-\tfrac{\lambda_1}{\lambda_2})\bar{x}, \tfrac{\mu_1}{\mu_2}(\bar{y}-\mu_2 y) + (1-\tfrac{\mu_1}{\mu_2})\bar{y}) \le$$

$$\le \tfrac{\lambda_1\mu_1}{\lambda_2\mu_2}(f(\bar{x}+\lambda_2 x,\bar{y}+\mu_2 y) - 2f(\bar{x},\bar{y}) + f(\bar{x}-\lambda_2 x,\bar{y}-\mu_2 y)) +$$

$$+ \tfrac{\lambda_1}{\lambda_2}(1-\tfrac{\mu_1}{\mu_2})(f(\bar{x}+\lambda_2 x,\bar{y}) - 2f(\bar{x},\bar{y}) + f(\bar{x}-\lambda_2 x,\bar{y})) +$$

$$+ \tfrac{\mu_1}{\mu_2}(1-\tfrac{\lambda_1}{\lambda_2})(f(\bar{x},\bar{y}+\mu_2 y) - 2f(\bar{x},\bar{y}) + f(\bar{x},\bar{y}-\mu_2 y))$$

при $(x,y) \in X \times Y$. Если бивыпуклая функция $f : X \times Y \to R$ является раздельно билипшицевой с постоянной L в окрестности точки $(\bar{x},\bar{y})$, то $f(\bar{x}+x,\bar{y}) - 2f(\bar{x},\bar{y}) + f(\bar{x}-x,\bar{y}) = 0$, $f(\bar{x},\bar{y}+y) - 2f(\bar{x},\bar{y}) + f(\bar{x},\bar{y}-y) = 0$ при $x \in X$ и $y \in Y$. Поэтому

$$\tfrac{1}{\lambda_1\mu_1}(f(\bar{x}+\lambda_1 x,\bar{y}+\mu_1 y) - 2f(\bar{x},\bar{y}) + f(\bar{x}-\lambda_1 x,\bar{y}-\mu_1 y)) \le$$

$$\le \tfrac{1}{\lambda_2\mu_2}(f(\bar{x}+\lambda_2 x,\bar{y}+\mu_2 y) - 2f(\bar{x},\bar{y}) + f(\bar{x}-\lambda_2 x,\bar{y}-\mu_2 y))$$

при $x \in X$ и $y \in Y$. Отсюда следует, что предел

$$f^{(2)}(\bar{x},\bar{y};x,y) = \lim_{\lambda \downarrow 0, \mu \downarrow 0} \tfrac{1}{\lambda\mu}(f(\bar{x}+\lambda x,\bar{y}+\mu y) - 2f(\bar{x},\bar{y}) + f(\bar{x}-\lambda x,\bar{y}-\mu y))$$

существует при $(x,y)\in X\times Y$, $(x,y)\to f^{(2)}(\overline{x},\overline{y};x,y)$ бисублинейная четная функция и $f^{(2)}(\overline{x},\overline{y};x,y)=f^{(2)+}(\overline{x},\overline{y};x,y)$ при $(x,y)\in X\times Y$. Отметим, что $\lambda\downarrow 0,\mu\downarrow 0$ означает, что $(0,0)<(\lambda_k,\mu_k)\le(\lambda_{k-1},\mu_{k-1})$ и $(\lambda_k,\mu_k)\to(0,0)$. Более того

$$f^{(2)}(\overline{x},\overline{y};x,y)=\inf_{\lambda>0,\mu>0}\frac{1}{\lambda\mu}(f(\overline{x}+\lambda x,\overline{y}+\mu y)-2f(\overline{x},\overline{y})+f(\overline{x}-\lambda x,\overline{y}-\mu y))\le$$
$$\le f(\overline{x}+x,\overline{y}+y)-2f(\overline{x},\overline{y})+f(\overline{x}-x,\overline{y}-y)$$

при $(x,y)\in X\times Y$. Ясно, что $f^{(2)}(\overline{x},\overline{y};0,y)=f^{(2)}(\overline{x},\overline{y};x,0)=f^{(2)}(\overline{x},\overline{y};0,0)=0$ при $(x,y)\in X\times Y$.

Лемма 2.7. Если бивыпуклая функция $f:X\times Y\to R$ является раздельно билипшицевой с постоянной L в окрестности точки $(\overline{x},\overline{y})$, то $\partial^2 f(\overline{x},\overline{y})=\overline{\partial}^2 f(\overline{x},\overline{y})$.

Если $A\subset B(X\times Y,R)$, то $s-$оболочкой множества A назовем множество

$$sA=\{x^*\in B(X\times Y,R):\sup\{\widetilde{b}(x,y):\widetilde{b}\in A\}\ge x^*(x,y)\ \text{ при }\ x\in X,\ y\in Y\}.$$

Отметим, что если $A_1,A_2\subset B(X\times Y,R)$ и $\sup\limits_{x_1^*\in A_1}x_1^*(x,y)\le\sup\limits_{x_2^*\in A_2)}x_2^*(x,y)$ при $(x,y)\in X\times Y$, то $sA_1\subset sA_2$. Ясно, что $A\subset sA$.

Теорема 2.1. Если $f_1:X\times Y\to R$ и $f_2:X\times Y\to R$ бивыпуклые раздельно билипшицевые с постоянными $L_1>0$, $L_2>0$ в окрестности точки $(\overline{x},\overline{y})$ функции, то f_1+f_2 также бивыпуклая раздельно билипшицевая с постоянной L_1+L_2 в окрестности точки $(\overline{x},\overline{y})$ функция,

$$(f_1+f_2)^{(2)}(\overline{x},\overline{y};x,y)=f_1^{(2)}(\overline{x},\overline{y};x,y)+f_2^{(2)}(\overline{x},\overline{y};x,y)$$

при $(x,y)\in X\times Y$ и

$$\partial^2(f_1+f_2)(\overline{x},\overline{y})=s(\partial^2 f_1(\overline{x},\overline{y})+\partial^2 f_2(\overline{x},\overline{y})).$$

Доказательство. Так как $f_1:X\times Y\to R$ и $f_2:X\times Y\to R$ бивыпуклые раздельно билипшицевые в окрестности точки $(\overline{x},\overline{y})$ функции, то существуют постоянные $L_1>0$, $L_2>0$ и $\delta>0$ такие, что

$$\left|f_1(\overline{x}+x,\overline{y}+y)-2f_1(\overline{x},\overline{y})+f_1(\overline{x}-x,\overline{y}-y)\right|\le L_1\|x\|\cdot\|y\|$$
$$\left|f_2(\overline{x}+x,\overline{y}+y)-2f_2(\overline{x},\overline{y})+f_2(\overline{x}-x,\overline{y}-y)\right|\le L_2\|x\|\cdot\|y\|$$

при $x\in\delta B_X$, $y\in\delta B_Y$. Поэтому

$$\left|f_1(\overline{x}+x,\overline{y}+y)-2f_1(\overline{x},\overline{y})+f_1(\overline{x}-x,\overline{y}-y)+f_2(\overline{x}+x,\overline{y}+y)-2f_2(\overline{x},\overline{y})+f_2(\overline{x}-x,\overline{y}-y\right|\le(L_1+L_2)\|x\|\cdot\|y\|$$

при $x\in\delta B_X$, $y\in\delta B_Y$. Также из определения имеем, что

$$(f_1+f_2)^{(2)}(\overline{x},\overline{y};x,y)=f_1^{(2)}(\overline{x},\overline{y};x,y)+f_2^{(2)}(\overline{x},\overline{y};x,y)$$

при $(x,y)\in X\times Y$. Поэтому из следствия 5.3.5 [11, с. 209] (см. также теорему 3.3)

$$(f_1+f_2)^{(2)}(\overline{x},\overline{y};x,y)=\sup_{x^*\in\partial^2(f_1+f_2)(\overline{x},\overline{y})}x^*(x,y)=f_1^{(2)}(\overline{x},\overline{y};x,y)+f_2^{(2)}(\overline{x},\overline{y};x,y)=$$
$$=\sup_{x_1^*\in\partial^2 f_1(\overline{x},\overline{y})}x_1^*(x,y)+\sup_{x_2^*\in\partial^2 f_2(\overline{x},\overline{y})}x_2^*(x,y)=\sup_{x^*\in\partial^2 f_1(\overline{x},\overline{y})+\partial^2 f_2(\overline{x},\overline{y})}x^*(x,y)$$

при $(x,y)\in X\times Y$. Отсюда следует, что $\partial^2(f_1+f_2)(\overline{x},\overline{y})=s(\partial^2 f_1(\overline{x},\overline{y})+\partial^2 f_2(\overline{x},\overline{y}))$. Теорема доказана.

Если T множество, $f_t:X\times Y\to R$ бивыпуклая функция при $t\in T$, то из предложения 2.2 [4, с. 27] следует, что $f(x,y)=\sup\limits_{t\in T}f_t(x,y)$ также бивыпуклая функция.

Лемма 2.8 доказана в [18, см. лемма 2.6].

Лемма 2.8. Пусть P компактное множество, $f_p:X\times Y\to R$ бивыпуклая функция при $p\in P$, существует $\delta>0$ такое, что функция $p\to f_p(x,y)$ полунепрерывна сверху при $(x,y)\in(x_0,y_0)+(\delta B_X\times\delta B_Y)$, $f(x,y)=\sup\limits_{p\in P}f_p(x,y)$, $P(x_0,y_0)=\{p\in P:f_p(x_0,y_0)=f(x_0,y_0)\}$. Тогда

$$f^{(2)+}((x_0,y_0);(x,y))\ge\sup_{p\in P(x_0,y_0)}f_p^{(2)+}((x_0,y_0);(x,y))$$

при $(x,y)\in X\times Y$ и $\operatorname{conv}\bigcup\limits_{p\in P(x_0,y_0)}\overline{\partial}^2 f_p(x_0,y_0)\subset\overline{\partial}^2 f(x_0,y_0)$.

Пусть $x^*\in B(X\times Y,R)$, $z^*\in(X\times Y)^*$, где $z^*=(x_1^*,y_1^*)$, $f:X\times Y\to R_{+\infty}$. Положим

$$f^*(x^*,z^*)=\sup_{(x,y)\in X\times Y}\{x^*(x,y)+z^*(x,y)-f(x,y)\}.$$

Лемма 2.9 доказана в [18, см. лемма 2.7].

Лемма 2.9. $(x^*, z^*) \in Df(\bar{x}, \bar{y})$ тогда и только тогда, когда

$$f^*(x^*, z^*) + f(\bar{x}, \bar{y}) = x^*(\bar{x}, \bar{y}) + z^*(\bar{x}, \bar{y}).$$

Пусть $g : X \times Y \to R_{+\infty}$ и $\operatorname{dom} g \neq \varnothing$. Так как для каждой пары $(x, y) \in X \times Y$, отображение $(x^*, z^*) \to x^*(x, y) + z^*(x, y)$ линейный функционал на $B(X \times Y, R) \times (X \times Y)^*$, то из определения $g^*(x^*, z^*)$ вытекает, что $(x^*, z^*) \to g^*(x^*, z^*)$ выпуклая функция. Положим

$$g^{**}(x, y) = \sup_{(x^*, z^*) \in B(X \times Y, R) \times (X \times Y)^*} \{x^*(x, y) + z^*(x, y) - g^*(x^*, z^*)\}.$$

Ясно, что $(x, y) \to x^*(x, y) + z^*(x, y) - g^*(x^*, z^*)$ 2-аффинная функция. Поэтому из предложения 1.3.1 [17, с.24] вытекает, что $(x, y) \to g^{**}(x, y)$ бивыпуклая функция. Из определения $g^*(x^*, z^*)$ получим, что $g^*(x^*, z^*) \geq x^*(x, y) + z^*(x, y) - g(x, y)$ при $(x, y) \in X \times Y$ и $(x^*, z^*) \in B(X \times Y, R) \times (X \times Y)^*$.

Поэтому $g^{**}(x, y) \leq g(x, y)$ для любого $(x, y) \in X \times Y$.

Если $(\nu, x, y) \in X \otimes Y \times X \times Y$, то положим

$$\tilde{g}^{**}(\nu, x, y) = \sup_{(x^*, z^*) \in B(X \times Y, R) \times (X \times Y)^*} \{u(\nu) + z^*(x, y) - g^*(x^*, z^*)\},$$

где $u(\nu) = \sum_{i=1}^{k} x^*(x^i, y^i)$ при $\nu = \sum_{i=1}^{k} x^i \otimes y^i$. Ясно, что $\tilde{g}^{**}(x \otimes y, x, y) = g^{**}(x, y)$ при $(x, y) \in X \times Y$. Из предложения 1.5 и 2.2 [4] следует, что $\tilde{g}^{**} : X \otimes Y \times X \times Y \to \overline{R}$ полунепрерывная снизу выпуклая функция.

Лемма 2.10. Если $f : X \times Y \to R_{+\infty}$ бивыпуклая функция, то $f^*(x^*, z^*) = f^{***}(x^*, z^*)$ при $(x^*, z^*) \in B(X \times Y, R) \times (X \times Y)^*$.

Доказательство. Так $f^{**}(x, y) \leq f(x, y)$ при $(x, y) \in X \times Y$, то из определения $f^{***}(x^*, z^*)$ следует, что $f^*(x^*, z^*) \leq f^{***}(x^*, z^*)$ при $(x^*, z^*) \in B(X \times Y, R) \times (X \times Y)^*$. Из определения $f^{**}(x, y)$ следует, что $f^{**}(x, y) \geq x^*(x, y) + z^*(x, y) - f^*(x^*, z^*)$ при $(x, y) \in X \times Y$ и $(x^*, z^*) \in B(X \times Y, R) \times (X \times Y)^*$. Поэтому

$$f^{***}(x^*, z^*) = \sup_{(x, y) \in X \times Y} \{x^*(x, y) + z^*(x, y) - f^{**}(x, y)\} \leq f^*(x^*, z^*)$$

при $(x^*, z^*) \in B(X \times Y, R) \times (X \times Y)^*$. Тогда получим, что $f^*(x^*, z^*) = f^{***}(x^*, z^*)$ при $(x^*, z^*) \in B(X \times Y, R) \times (X \times Y)^*$. Лемма доказана.

Из леммы 2.10 следует, что $f^{****}(x, y) = f^{**}(x, y)$ при $(x, y) \in X \times Y$.

Лемма 2.11. Если $Df(\bar{x}, \bar{y}) \neq \varnothing$, то $f^{**}(\bar{x}, \bar{y}) = f(\bar{x}, \bar{y})$.

Доказательство. Из леммы 2.9 следует, что $(x^*, z^*) \in Df(\bar{x}, \bar{y})$ тогда и только тогда, когда $f^*(x^*, z^*) + f(\bar{x}, \bar{y}) = x^*(\bar{x}, \bar{y}) + z^*(\bar{x}, \bar{y})$. Так как

$$f^{**}(x, y) = \sup_{(x^*, z^*) \in B(X \times Y, R) \times (X \times Y)^*} \{x^*(x, y) + z^*(x, y) - f^*(x^*, z^*)\},$$

то

$$f^{**}(\bar{x}, \bar{y}) \geq x^*(\bar{x}, \bar{y}) + z^*(\bar{x}, \bar{y}) - x^*(\bar{x}, \bar{y}) - z^*(\bar{x}, \bar{y}) + f(\bar{x}, \bar{y}) = f(\bar{x}, \bar{y}),$$

т.е. $f^{**}(\bar{x}, \bar{y}) \geq f(\bar{x}, \bar{y})$. Так как $f^{**}(\bar{x}, \bar{y}) \leq f(\bar{x}, \bar{y})$, то имеем, что $f^{**}(\bar{x}, \bar{y}) = f(\bar{x}, \bar{y})$. Лемма доказана.

Лемма 2.12. $(\bar{x}, \bar{y}) \in \partial f^*(\bar{x}^*, \bar{z}^*)$ тогда и только тогда, когда

$$f^*(\bar{x}^*, \bar{z}^*) + f^{**}(\bar{x}, \bar{y}) = \bar{x}^*(\bar{x}, \bar{y}) + \bar{z}^*(\bar{x}, \bar{y}).$$

Доказательство. Если $(\bar{x}, \bar{y}) \in \partial f^*(x^*, z^*)$, то по определению имеем, что

$$f^*(x^*, z^*) - f^*(\bar{x}^*, \bar{z}^*) \geq x^*(\bar{x}, \bar{y}) + z^*(\bar{x}, \bar{y}) - \bar{x}^*(\bar{x}, \bar{y}) - \bar{z}^*(\bar{x}, \bar{y})$$

при $(x^*, z^*) \in B(X \times Y, R) \times (X \times Y)^*$. Отсюда следует, что

$$\bar{x}^*(\bar{x}, \bar{y}) + \bar{z}^*(\bar{x}, \bar{y}) - f^*(\bar{x}^*, \bar{z}^*) \geq x^*(\bar{x}, \bar{y}) + z^*(\bar{x}, \bar{y}) - f^*(x^*, z^*).$$

Поэтому $\bar{x}^*(\bar{x}, \bar{y}) + \bar{z}^*(\bar{x}, \bar{y}) - f^*(\bar{x}^*, \bar{z}^*) = f^{**}(\bar{x}, \bar{y})$. Обратно, если $\bar{x}^*(\bar{x}, \bar{y}) + \bar{z}^*(\bar{x}, \bar{y}) - f^*(\bar{x}^*, \bar{z}^*) = f^{**}(\bar{x}, \bar{y})$,

то $\bar{x}^*(\bar{x},\bar{y})+\bar{z}^*(\bar{x},\bar{y})-f^*(\bar{x}^*,\bar{z}^*)=\sup\limits_{(x^*,z^*)\in B(X\times Y,R)\times(X\times Y)^*}\{x^*(\bar{x},\bar{y})+z^*(\bar{x},\bar{y})-f^*(x^*,z^*)\}$.

Отсюда следует, что

$$f^*(x^*,z^*)-f^*(\bar{x}^*,\bar{z}^*)\ge x^*(\bar{x},\bar{y})+z^*(\bar{x},\bar{y})-\bar{x}^*(\bar{x},\bar{y})-\bar{z}^*(\bar{x},\bar{y})$$

при $(x^*,z^*)\in B(X\times Y,R)\times(X\times Y)^*$, т.е. $(\bar{x},\bar{y})\in\partial f^*(\bar{x}^*,\bar{z}^*)$. Лемма доказана.

Лемма 2.13. Если $g(\cdot)\in\Gamma_0(X\otimes Y\times X\times Y)$ и $f(x,y)=g(x\otimes y,x,y)$ при $x\in X$ и $y\in Y$, то $f^{**}(x,y)=f(x,y)$ для любого $(x,y)\in X\times Y$.

Доказательство. Пусть $u\in(X\otimes Y)^*=B(X\times Y,R)$, где $u(\nu)=\sum\limits_{i=1}^{k}x^*(x^i,y^i)$ при $\nu=\sum\limits_{i=1}^{k}x^i\otimes y^i$, $k\in N$, $x^*\in B(X\times Y,R)$. Положив

$$\tilde{g}(\nu,x,y)=\begin{cases}f(x,y):(\nu,x,y)=(x\otimes y,x,y)\in X\otimes Y\times X\times Y,\\ +\infty:\text{в других случаях}\end{cases}$$ имеем, что

$$\tilde{g}^*(x^*,z^*)=\sup_{(\nu,x,y)\in X\otimes Y\times X\times Y}\{u(\nu)+z^*(x,y)-\tilde{g}(\nu,x,y)\}=\sup_{x\in X,y\in Y}\{x^*(x,y)+z^*(x,y)-f(x,y)\}=$$

$$=f^*(x^*,z^*)\le\sup_{(\nu,x,y)\in X\otimes Y\times X\times Y}\{u(\nu)+z^*(x,y)-g(\nu,x,y)\}=g^*(x^*,z^*)$$

при $(x^*,z^*)\in B(X\times Y,R)\times(X\times Y)^*$. Поэтому

$$f^{**}(x,y)=\sup_{(x^*,z^*)\in B(X\times Y,R)\times(X\times Y)^*}\{x^*(x,y)+z^*(x,y)-f^*(x^*,z^*)\}\ge$$

$$\ge g^{**}(x\otimes y,x,y)=\sup_{(x^*,z^*)\in B(X\times Y,R)\times(X\times Y)^*}\{x^*(x,y)+z^*(x,y)-g^*(x^*,z^*)\}.$$

Получим, что $f(x,y)\ge f^{**}(x,y)\ge g^{**}(x\otimes y,x,y)=g(x\otimes y,x,y)=f(x,y)$ при $x\in X$ и $y\in Y$. Отсюда следует, что $f^{**}(x,y)=f(x,y)$ при $(x,y)\in X\times Y$. Лемма доказана.

Если $f\in A(X\times Y)$, то из леммы 2.13 следует, что $f^{**}(x,y)=f(x,y)$ для любого $(x,y)\in X\times Y$.

Отметим, что если $f\in A(X\times Y)$, то существует множество $S\subset B(X\times Y,R)\times X^*\times Y^*\times R$ такое, что $f(x,y)=\sup\limits_{(x^*,v^*,y^*,\alpha)\in S}\{x^*(x,y)+v^*(x)+y^*(y)+\alpha\}$. Поэтому

$$f^*(x^*,v^*,y^*)=\sup_{(x,y)\in X\times Y}\{x^*(x,y)+v^*(x)+y^*(y)-\sup_{(\tilde{x}^*,\tilde{v}^*,\tilde{y}^*,\alpha)\in S}\{\tilde{x}^*(x,y)+\tilde{v}^*(x)+\tilde{y}^*(y)+\alpha\}\}=$$

$$=\sup_{(x,y)\in X\times Y}\inf_{(\tilde{x}^*,\tilde{v}^*,\tilde{y}^*,\alpha)\in S}\{x^*(x,y)+z^*(x,y)-\tilde{x}^*(x,y)-\tilde{v}^*(x)-\tilde{y}^*(y)-\alpha\}=$$

$$=\begin{cases}\inf\limits_{(x^*,v^*,y^*,\alpha)\in S}\{-\alpha\}:\exists\alpha\in R,\ \text{что}\ (x^*,v^*,y^*,\alpha)\in S,\\ +\infty:\forall\alpha\in R,\ \text{что}\ (x^*,v^*,y^*,\alpha)\notin S.\end{cases}$$

Тогда

$$f^{**}(x,y)=\sup_{(x^*,v^*,y^*)\in B(X\times Y,R)\times(X\times Y)^*}\{x^*(x,y)+v^*(x)+y^*(y)-f^*(x^*,v^*,y^*)\}=$$

$$=\sup_{(x^*,v^*,y^*,\alpha)\in S}\{x^*(x,y)+v^*(x)+y^*(y)-\inf_{(x^*,v^*,y^*,\alpha)\in S}\{-\alpha\}\}=$$

$$=\sup_{(x^*,v^*,y^*,\alpha)\in S}\{x^*(x,y)+v^*(x)+y^*(y)+\alpha\}=f(x,y).$$

Отсюда также следует, что $f^{**}(x,y)=f(x,y)$ при $(x,y)\in X\times Y$.

Из соотношения $f^{**}(x,y)=\sup\limits_{(x^*,z^*)\in B(X\times Y,R)\times(X\times Y)^*}\{x^*(x,y)+z^*(x,y)-f^*(x^*,z^*)\}$ следует, что если f^{**} собственная функция, то $f^{**}\in A(X\times Y)$. Поэтому справедливо следующее следствие.

Следствие 2.1. Если f собственная функция, то $f^{**}(x,y)=f(x,y)$ при $(x,y)\in X\times Y$ тогда и только тогда, когда $f\in A(X\times Y)$.

Теорема 2.2. Если $q:X\times Y\to R_{+\infty}$, то $(\bar{x}^*,\bar{z}^*)\in Dq(\bar{x},\bar{y})$ тогда и только тогда, когда $(\bar{u}(\cdot),\bar{z}^*)\in\partial\tilde{q}^{**}(\bar{x}\otimes\bar{y},\bar{x},\bar{y})$ и $\tilde{q}^{**}(\bar{x}\otimes\bar{y},\bar{x},\bar{y})=q(\bar{x},\bar{y})$, где $\bar{u}(\nu)=\sum\limits_{i=1}^{k}\bar{x}^*(x^i,y^i)$ при $\nu=\sum\limits_{i=1}^{k}x^i\otimes y^i$, $k\in N$.

Доказательство. Пусть $\tilde{q}^{**}(\bar{x}\otimes\bar{y},\bar{x},\bar{y})=q(\bar{x},\bar{y})$ и

$(\bar{u}(\cdot),\bar{z}^*)\in(X\otimes Y)^*\times(X\times Y)^*=B(X\times Y,R)\times(X\times Y)^*$ принадлежит $\partial\tilde{q}^{**}(\bar{x}\otimes\bar{y},\bar{x},\bar{y})$. Тогда

$$\tilde{q}^{**}(\nu,x,y)-\tilde{q}^{**}(\bar{x}\otimes\bar{y},\bar{x},\bar{y})\geq\bar{u}^*(\nu)-\bar{u}^*(\bar{x}\otimes\bar{y})+\bar{z}^*(x,y)-\bar{z}^*(\bar{x},\bar{y})$$

при $(\nu,x,y)\in X\otimes Y\times X\times Y$. Если учесть, что $q(x,y)\geq q^{**}(x,y)=\tilde{q}^{**}(x\otimes y,x,y),$ то по условию получим

$$q(x,y)-q(\bar{x},\bar{y})\geq\bar{u}^*(x,y)-\bar{u}^*(\bar{x}\otimes\bar{y})+\bar{z}^*(x,y)-\bar{z}^*(\bar{x},\bar{y})$$

при $(x,y)\in X\times Y$. Поэтому $(\bar{x}^*,\bar{z}^*)\in Dq(\bar{x},\bar{y}),$ где $\bar{x}^*(x,y)=\bar{u}^*(x\otimes y)$ при $(x,y)\in X\times Y$. Обратно, если $(\bar{x}^*,\bar{z}^*)\in Dq(\bar{x},\bar{y}),$ то $q(x,y)-q(\bar{x},\bar{y})\geq\bar{x}^*(x,y)+\bar{z}^*(x,y)-\bar{x}^*(\bar{x},\bar{y})-\bar{z}^*(\bar{x},\bar{y})$ при $(x,y)\in X\times Y$. Отсюда следует, что

$$\sup_{(x,y)\in X\times Y}\{x^*(x,y)+z^*(x,y)-q(x,y)\}+q(\bar{x},\bar{y})\leq\sup_{(x,y)\in X\times Y}\{x^*(x,y)+z^*(x,y)-\bar{x}^*(x,y)-$$

$$-\bar{z}^*(x,y)\}+\bar{x}^*(\bar{x},\bar{y})+\bar{z}^*(\bar{x},\bar{y}).$$

Поэтому $q^*(x^*,z^*)+q(\bar{x},\bar{y})\leq\bar{x}^*(\bar{x},\bar{y})+\bar{z}^*(\bar{x},\bar{y})+\begin{cases}0:(x^*,z^*)=(\bar{x}^*,\bar{z}^*),\\+\infty:(x^*,z^*)\neq(\bar{x}^*,\bar{z}^*)\end{cases}$ при $(x^*,z^*)\in B(X\times Y,R)\times(X\times Y)^*$. Действительно, если $x^*\neq\bar{x}^*$, то существует точка $(x_1,y_1)\in X\times Y$ такая, что $x^*(x_1,y_1)-\bar{x}^*(x_1,y_1)>0$. Поэтому

$$\sup_{(x,y)\in X\times Y}\{x^*(x,y)+z^*(x,y)-\bar{x}^*(x,y)-\bar{z}^*(x,y)\}\geq\sup_{\lambda>0}\{x^*(\lambda x_1,\lambda y_1)+z^*(\lambda x_1,\lambda y_1)-$$

$$-\bar{x}^*(\lambda x_1,\lambda y_1)-\bar{z}^*(\lambda x_1,\lambda y_1)\}=\sup_{\lambda>0}\lambda^2\{x^*(x_1,y_1)-\bar{x}^*(x_1,y_1)+\tfrac{1}{\lambda}(z^*(x_1,y_1)-\bar{z}^*(x_1,y_1))\}=+\infty.$$

Если $x^*=\bar{x}^*$ и $z^*\neq\bar{z}^*$, то существует точка $(x_1,y_1)\in X\times Y$ такая, что $z^*(x_1,y_1)-\bar{z}^*(x_1,y_1)>0$. Поэтому

$$\sup_{(x,y)\in X\times Y}\{x^*(x,y)+z^*(x,y)-\bar{x}^*(x,y)-\bar{z}^*(x,y)\}\geq\sup_{\lambda>0}\{x^*(\lambda x_1,\lambda y_1)+z^*(\lambda x_1,\lambda y_1)-$$

$$-\bar{x}^*(\lambda x_1,\lambda y_1)-\bar{z}^*(\lambda x_1,\lambda y_1)\}=\sup_{\lambda>0}\lambda\{(z^*(x_1,y_1)-\bar{z}^*(x_1,y_1))\}=+\infty.$$

Если $\nu\in X\otimes Y$, $\nu=\sum_{i=1}^{k}x^i\otimes y^i$, то положив $u(\nu)=\sum_{i=1}^{k}x^*(x^i,y^i)$ и $\bar{u}(\nu)=\sum_{i=1}^{k}\bar{x}^*(x^i,y^i)$ имеем, что

$$\sup_{(x^*,z^*)\in B(X\times Y,R)\times(X\times Y)^*}\{u(\nu)+z^*(x,y)-q^*(x^*,z^*)-q(\bar{x},\bar{y})\}\geq$$

$$\geq\bar{u}(\nu)+\bar{z}^*(x,y)-\bar{x}^*(\bar{x},\bar{y})-\bar{z}^*(\bar{x},\bar{y}),$$

т.е. $\tilde{q}^{**}(\nu,x,y)-q(\bar{x},\bar{y})\geq\bar{u}(\nu)+\bar{z}^*(x,y)-\bar{x}^*(\bar{x},\bar{y})-\bar{z}^*(\bar{x},\bar{y})$ при $(\nu,x,y)\in X\otimes Y\times X\times Y$. Взяв $(\nu,x,y)=(\bar{x}\otimes\bar{y},\bar{x},\bar{y})$, получим

$$\tilde{q}^{**}(\bar{x}\otimes\bar{y},\bar{x},\bar{y})-q(\bar{x},\bar{y})\geq\bar{u}(\bar{x}\otimes\bar{y})-\bar{z}^*(\bar{x},\bar{y})-\bar{x}^*(\bar{x},\bar{y})-\bar{z}^*(\bar{x},\bar{y})=0,$$

т.е. $\tilde{q}^{**}(\bar{x}\otimes\bar{y},\bar{x},\bar{y})\geq q(\bar{x},\bar{y})$. Так как $\tilde{q}^{**}(x\otimes y,x,y)=q^{**}(x,y)$ и $q^{**}(x,y)\leq q(x,y)$ при $(x,y)\in X\times Y$, то получим, что $\tilde{q}^{**}(\bar{x}\otimes\bar{y},\bar{x},\bar{y})\leq q(\bar{x},\bar{y})$. Поэтому $\tilde{q}^{**}(\bar{x}\otimes\bar{y},\bar{x},\bar{y})=q(\bar{x},\bar{y})$. Тогда $\tilde{q}^{**}(\nu,x,y)-\tilde{q}^{**}(\bar{x}\otimes\bar{y},\bar{x},\bar{y})\geq\bar{u}(\nu)+\bar{z}^*(x,y)-\bar{u}^*(\bar{x}\otimes\bar{y})-\bar{z}^*(\bar{x},\bar{y})$ при $(\nu,x,y)\in X\otimes Y\times X\times Y$, т.е. $(\bar{u}(\cdot),\bar{z}^*)\in\partial\tilde{q}^{**}(\bar{x}\otimes\bar{y},\bar{x},\bar{y})$. Теорема доказана.

Если $z^*\in(X\times Y)^*$, то положив

$\partial_2 f_{z^*}(\bar{x},\bar{y})=\{x^*\in B(X\times Y,R):f(x,y)-f(\bar{x},\bar{y})\geq x^*(x,y)+z^*(x,y)-x^*(\bar{x},\bar{y})-z^*(\bar{x},\bar{y})$ при $(x,y)\in X\times Y\}$

получим, что $D_2f(\bar{x},\bar{y})=\bigcup_{z^*\in(X\times Y)^*}\partial_2 f_{z^*}(\bar{x},\bar{y})$.

Пусть X и Y банаховы пространства, $f:X\times Y\to R_{+\infty}$ бивыпуклая функция.

Теорема 2.3. Пусть $(\bar{x},\bar{y})\in\operatorname{dom}f$. Для того чтобы $(\bar{x},\bar{y})\in X\times Y$ была точкой минимума бивыпуклой функции $f:X\times Y\to R_{+\infty}$ во всем пространстве, необходимо и достаточно, чтобы $(0,0,0)\in Df(\bar{x},\bar{y})$.

Доказательство. Если $f(\bar{x},\bar{y})=\min_{(x,y)\in X\times Y}f(x,y)$, то $f(\bar{x},\bar{y})\leq f(x,y)$ при $(x,y)\in X\times Y$. Поэтому $f(x,y)-f(\bar{x},\bar{y})\geq 0$ при $(x,y)\in X\times Y$. Отсюда следует, что $(0,0,0)\in Df(\bar{x},\bar{y})$.

Обратно, если $(0,0,0)\in Df(\bar{x},\bar{y})$, то $f(x,y)-f(\bar{x},\bar{y})\geq 0$ при $(x,y)\in X\times Y$. Отсюда следует, что $f(\bar{x},\bar{y})=\min_{(x,y)\in X\times Y}f(x,y)$. Теорема доказана.

Если $(\bar{x},\bar{y}) \in \operatorname{dom} f$ и $f(\bar{x},\bar{y}) = \min\limits_{(x,y)\in X\times Y} f(x,y)$, то получим, что

$$f(\bar{x},\bar{y}) = -\max_{(x,y)\in X\times Y}\{-f(x,y)\} = -f^*(0,0,0).$$

Получим, что $f(\bar{x},\bar{y}) + f^*(0,0,0) = 0$. Отсюда также следует, что $(0,0,0) \in Df(\bar{x},\bar{y})$.

Следствие 2.2. Если бивыпуклая функция f достигает на $X\times Y$ в точке $(\bar{x},\bar{y})$ минимума, то $(0,0,0) \in Df(\bar{x},\bar{y})$ и $(\bar{x},\bar{y}) \in \partial f^*(0,0,0)$.

Доказательство. Из теоремы 2.2 следует, что $(0,0,0) \in Df(\bar{x},\bar{y})$. Так как $Df(\bar{x},\bar{y}) \neq \varnothing$, то из леммы 2.11 следует, что $f^{**}(\bar{x},\bar{y}) = f(\bar{x},\bar{y})$. Поэтому из равенства $f(\bar{x},\bar{y}) + f^*(0,0,0) = 0$ следует, что $f^{**}(\bar{x},\bar{y}) + f^*(0,0,0) = 0$. Тогда из леммы 2.12 следует, что $(\bar{x},\bar{y}) \in \partial f^*(0,0,0)$. Следствие доказано.

Пусть X и Y банаховы пространства, $C \subset X\times Y$ и $(\bar{x},\bar{y}) \in C$. Положим

$$\delta_C(x,y) = \begin{cases} 0; & (x,y) \in C, \\ +\infty; & (x,y) \notin C. \end{cases}$$

Множество $D\delta_C(\bar{x},\bar{y})$ назовем бинормальным конусом к C в точке $(\bar{x},\bar{y})$ и обозначим через $\Omega_C(\bar{x},\bar{y})$. Из определения $D\delta_C(\bar{x},\bar{y})$ имеем

$$\Omega_C(\bar{x},\bar{y}) = \{(x^*,v^*,y^*) \in B(X\times Y,R)\times X^*\times Y^*:\ x^*(x,y) + v^*(x) + y^*(y) - x^*(\bar{x},\bar{y}) - v^*(\bar{x}) - y^*(\bar{y}) \le 0,\ \forall (x,y) \in C\}.$$

Из определения $\Omega_C(\bar{x},\bar{y})$ также следует, что $(x^*,v^*,y^*) \in \Omega_C(\bar{x},\bar{y})$ тогда и только тогда, когда $\sup\{x^*(x,y) + v^*(x) + y^*(y) : (x,y) \in C\} = x^*(\bar{x},\bar{y}) + v^*(\bar{x}) + y^*(\bar{y})$.

Если $A \subset B(X\times Y,R)\times X^*\times Y^*$, то положим $\operatorname{con} A = \{\lambda(x^*,v^*,y^*) : (x^*,v^*,y^*) \in A,\ \lambda \ge 0\}$.

Лемма 2.14. Если $g : X\times Y \to R$, $C = \{(x,y) \in X\times Y : g(x,y) \le g(\bar{x},\bar{y})\}$, то $\operatorname{con} Dg(\bar{x},\bar{y}) \subset \Omega_C(\bar{x},\bar{y})$.

Доказательство. Если $(x^*,v^*,y^*) \in Dg(\bar{x},\bar{y})$, то $g(x,y) - g(\bar{x},\bar{y}) \ge x^*(x,y) + v^*(x) + y^*(y) - x^*(\bar{x},\bar{y}) - v^*(\bar{x}) - y^*(\bar{y})$ при $(x,y) \in X\times Y$. Поэтому $0 \ge g(x,y) - g(\bar{x},\bar{y}) \ge x^*(x,y) + v^*(x) + y^*(y) - x^*(\bar{x},\bar{y}) - v^*(\bar{x}) - y^*(\bar{y})$ при $(x,y) \in C$, т.е. $x^*(x,y) + v^*(x) + y^*(y) - x^*(\bar{x},\bar{y}) - v^*(\bar{x}) - y^*(\bar{y}) \le 0$ при $(x,y) \in C$. Получим, что $Dg(\bar{x},\bar{y}) \subset \Omega_C(\bar{x},\bar{y})$. Пусть $\lambda \ge 0$ и $(x^*,v^*,y^*) \in Dg(\bar{x},\bar{y})$. Тогда $\lambda(x^*(x,y) + v^*(x) + y^*(y) - x^*(\bar{x},\bar{y}) - v^*(\bar{x}) - y^*(\bar{y})) \le 0$ при $(x,y) \in C$, т.е. $\operatorname{con} D\, g(\bar{x},\bar{y}) \subset \Omega_C(\bar{x},\bar{y})$. Лемма доказана.

Положим

$$\partial^{2\div} f(\bar{x},\bar{y}) = \{x^* \in B(X\times Y,R) : f(\bar{x}+x,\bar{y}+y) + f(\bar{x}-x,\bar{y}) + f(\bar{x},\bar{y}-y) - 3f(\bar{x},\bar{y}) \ge x^*(x,y) \text{ при } (x,y) \in X\times Y\}.$$

Лемма 2.15. Если $(x^*,x_1^*,y_1^*) \in Df(\bar{x},\bar{y})$, то

$$f(\bar{x}+x,\bar{y}+y) + f(\bar{x}-x,\bar{y}) + f(\bar{x},\bar{y}-y) - 3f(\bar{x},\bar{y}) \ge x^*(x,y)$$

при $(x,y) \in X\times Y$, т.е. $x^* \in \partial^{2\div} f(\bar{x},\bar{y})$.

Доказательство. Если $(x^*,x_1^*,y_1^*) \in Df(\bar{x},\bar{y})$, то из определения $Df(\bar{x},\bar{y})$ вытекает, что

$$f(\bar{x}+x,\bar{y}+y) - f(\bar{x},\bar{y}) \ge x^*(\bar{x}+x,\bar{y}+y) - x^*(\bar{x},\bar{y}) + x_1^*(\bar{x}+x) - x_1^*(\bar{x}) + y_1^*(\bar{y}+y) - y_1^*(\bar{y}),$$

$$f(\bar{x}-x,\bar{y}) - f(\bar{x},\bar{y}) \ge x^*(\bar{x}-x,\bar{y}) - x^*(\bar{x},\bar{y}) + x_1^*(\bar{x}-x) - x_1^*(\bar{x}) + y_1^*(\bar{y}) - y_1^*(\bar{y}),$$

$$f(\bar{x},\bar{y}-y) - f(\bar{x},\bar{y}) \ge x^*(\bar{x},\bar{y}-y) - x^*(\bar{x},\bar{y}) + x_1^*(\bar{x}) - x_1^*(\bar{x}) + y_1^*(\bar{y}-y) - y_1^*(\bar{y})$$

при $(x,y) \in X\times Y$. Сложив эти соотношения имеем

$$f(\bar{x}+x,\bar{y}+y) + f(\bar{x}-x,\bar{y}) + f(\bar{x},\bar{y}-y) - 3f(\bar{x},\bar{y}) \ge x^*(x,y)$$

при $(x,y) \in X\times Y$. Отсюда следует, что $x^* \in \partial^{2\div} f(\bar{x},\bar{y})$. Лемма доказана.

Если $g : X\times Y \to R_{+\infty}$, $(\bar{x},\bar{y}) \in \operatorname{dom} f$ и $(x,y) \in X\times Y$, то положим

$$f^{(2)\div}((\bar{x},\bar{y});(x,y)) = \limsup_{\lambda\downarrow 0,\mu\downarrow 0} \frac{1}{\lambda\mu}(f(\bar{x}+\lambda x,\bar{y}+\mu y) + f(\bar{x}-\lambda x,\bar{y}) + f(\bar{x},\bar{y}-\mu y) - 3f(\bar{x},\bar{y})),$$

$$\overline{\partial}^{2\div} f(\bar{x},\bar{y}) = \{x^* \in B(X\times Y,R) :\ f^{(2)\div}((\bar{x},\bar{y});(x,y)) \ge x^*(x,y) \text{ при } (x,y) \in X\times Y\}.$$

Легко проверяется, что $x \to f^{(2)\div}((\bar{x},\bar{y});(x,y))$ и $y \to f^{(2)\div}((\bar{x},\bar{y});(x,y))$ выпуклые положительно однородные функции.

Функция $f: X\times Y\to R$ называется раздельно 2-липшицевой с постоянной L в окрестности точки $(\overline{x},\overline{y})$, если существует число $\delta>0$ такое, что

$$\left|f(\overline{x}+x,\overline{y}+y)+f(\overline{x}-x,\overline{y})+f(\overline{x},\overline{y}-y)-3f(\overline{x},\overline{y})\right|\le L\|x\|\cdot\|y\|$$

при $x\in\delta B_X$, $y\in\delta B_Y$.

Если $f: X\times Y\to R$ раздельно 2-липшицевая функция с постоянной L в окрестности точки $(\overline{x},\overline{y})$, то $f(\overline{x}+x,\overline{y})-2f(\overline{x},\overline{y})+f(\overline{x}-x,\overline{y})=0$, $f(\overline{x},\overline{y}+y)-2f(\overline{x},\overline{y})+f(\overline{x},\overline{y}-y)=0$ при $x\in X$ и $y\in Y$, т.е. свойство локально раздельно билипшицевости функции в окрестности точки $(\overline{x},\overline{y})$ выполняется при жестких условиях.

Если функции $f_1: X\to R$ и $f_2: Y\to R$ и $f(x,y)=f_1(x)f_2(y)$ при $x\in X$ и $y\in Y$, то

$$f(\overline{x}+x,\overline{y}+y)+f(\overline{x}-x,\overline{y})+f(\overline{x},\overline{y}-y)-3f(\overline{x},\overline{y})=f_1(\overline{x}+x)f_2(\overline{y}+y)+f_1(\overline{x}-x)f_2(\overline{y})+f_1(\overline{x})f_2(\overline{y}-y)-$$
$$-3f_1(\overline{x})f_2(\overline{y})=(f_1(\overline{x}+x)-f_1(\overline{x}))(f_2(\overline{y}+y)-f_2(\overline{y}))+f_2(\overline{y})(f_1(\overline{x}+x)-2f_1(\overline{x})+f_1(\overline{x}-x))+$$
$$+f_1(\overline{x})(f_2(\overline{y}+y)-2f_2(\overline{y})+f_2(\overline{y}-y))$$

при $x\in X$ и $y\in Y$. Если $\left|f_1(\overline{x}+x)-f_1(\overline{x})\right|\le L_1\|x\|$ при $x\in\delta B_X$, $\left|f_2(\overline{y}+y)-f_2(\overline{y})\right|\le L_2\|y\|$ при $y\in\delta B_Y$ и $f_1(\overline{x})=f_2(\overline{y})=0$, то отсюда следует, что

$$\left|f(\overline{x}+x,\overline{y}+y)+f(\overline{x}-x,\overline{y})+f(\overline{x},\overline{y}-y)-3f(\overline{x},\overline{y})\right|=\left|(f_1(\overline{x}+x)-f_1(\overline{x}))(f_2(\overline{y}+y)-f_2(\overline{y}))\right|=L_1L_1\|x\|\|y\|$$

при $x\in\delta B_X$, $y\in\delta B_Y$.

Пусть $(x_i,y_i)\in X\times Y$ при $i=0,1,\dots,m$. Если $(x_i^*,v_i^*,y_i^*)\in Df(x_i,y_i)$, то

$$f(x,y)-f(x_i,y_i)\ge x_i^*(x,y)+v_i^*(x)+y_i^*(y)-x_i^*(x_i,y_i)-v_i^*(x_i)-y_i^*(y_i)$$

при $(x,y)\in X\times Y$. Положив $(x,y)=(x_{i+1},y_{i+1})$ и $(x_{m+1},y_{m+1})=(x_0,y_0)$, отсюда имеем, что

$$f(x_{i+1},y_{i+1})-f(x_i,y_i)\ge x_i^*(x_{i+1},y_{i+1})+v_i^*(x_{i+1})+y_i^*(y_{i+1})-x_i^*(x_i,y_i)-v_i^*(x_i)-y_i^*(y_i)$$

при $i=0,1,\dots,m$. Так как $(x_{m+1},y_{m+1})=(x_0,y_0)$, то

$$[f(x_1,y_1)-f(x_0,y_0)]+[f(x_2,y_2)-f(x_1,y_1)]+\dots+[f(x_m,y_m)-f(x_{m-1},y_{m-1})]+[f(x_0,y_0)-f(x_m,y_m)]=0.$$

Тогда получим, что

$$[x_0^*(x_1,y_1)+v_0^*(x_1)+y_0^*(y_1)-x_0^*(x_0,y_0)-v_0^*(x_0)-y_0^*(y_0)]+[x_1^*(x_2,y_2)+v_1^*(x_2)+y_1^*(y_2)-$$
$$-x_1^*(x_1,y_1)-v_1^*(x_1)-y_1^*(y_1)]+\dots+[x_{m-1}^*(x_m,y_m)+v_{m-1}^*(x_m)+y_{m-1}^*(y_m)-x_{m-1}^*(x_{m-1},y_{m-1})-$$
$$-v_{m-1}^*(x_{m-1})-y_{m-1}^*(y_{m-1})]+[x_m^*(x_0,y_0)+v_m^*(x_0)+y_m^*(y_0)-x_m^*(x_m,y_m)-v_m^*(x_m)-y_m^*(y_m)]\le 0.$$

Аналогично субдифференциала выпуклых функций (см. [6, с.254]), если f - собственная бивыпуклая функция, то многозначное отображение Df, действующее из $X\times Y$ в $B(X\times Y,R)\times X^*\times Y^*$ назовем циклически монотонным.

Определим функцию g формулой

$$g(x,y)=\sup\{x_m^*(x,y)+v_m^*(x)+y_m^*(y)-x_m^*(x_m,y_m)-v_m^*(x_m)-y_m^*(y_m)+x_{m-1}^*(x_m,y_m)+v_{m-1}^*(x_m)+$$
$$+y_{m-1}^*(y_m)-x_{m-1}^*(x_{m-1},y_{m-1})-v_{m-1}^*(x_{m-1})-y_{m-1}^*(y_{m-1})+\dots+x_1^*(x_2,y_2)+v_1^*(x_2)+y_1^*(y_2)-$$
$$-x_1^*(x_1,y_1)-v_1^*(x_1)-y_1^*(y_1)+x_0^*(x_1,y_1)+v_0^*(x_1)+y_0^*(y_1)-x_0^*(x_0,y_0)-v_0^*(x_0)-y_0^*(y_0)\},$$

где верхняя грань берется всевозможным конечным набором пар $(x_i^*,v_i^*,y_i^*)\in Df(x_i,y_i)$ при $i=1,\dots,m$. Так как g есть верхняя грань некоторого семейства 2-аффинных функций, то g полунепрерывная снизу бивыпуклая функция. Из циклической монотонности Df следует, что $g(x_0,y_0)=0$, т.е. g собственная функция.

Лемма 2.16. Если $(x^*,v^*,y^*)\in Df(\overline{x},\overline{y})$, то $(x^*,v^*,y^*)\in Dg(\overline{x},\overline{y})$.

Доказательство. Достаточно проверить, что неравенство

$$g(x,y)>\alpha+x^*(x,y)+v^*(x)+y^*(y)-x^*(\overline{x},\overline{y})-v^*(\overline{x})-y^*(\overline{y})$$

выполняется при $\alpha<g(\overline{x},\overline{y})$ и $(x,y)\in X\times Y$. Действительно, если $\alpha<g(\overline{x},\overline{y})$, то из определения g следует существование некоторого набора пар $(x_i^*,v_i^*,y_i^*)\in Df(x_i,y_i)$ при $i=1,\dots,m$ и

$$\alpha<x_m^*(\overline{x},\overline{y})+v_m^*(\overline{x})+y_m^*(\overline{y})-x_m^*(x_m,y_m)-v_m^*(x_m)-y_m^*(y_m)+x_{m-1}^*(x_m,y_m)+v_{m-1}^*(x_m)+$$
$$+y_{m-1}^*(y_m)-x_{m-1}^*(x_{m-1},y_{m-1})-v_{m-1}^*(x_{m-1})-y_{m-1}^*(y_{m-1})+\dots+x_1^*(x_2,y_2)+v_1^*(x_2)+y_1^*(y_2)-$$
$$-x_1^*(x_1,y_1)-v_1^*(x_1)-y_1^*(y_1)+x_0^*(x_1,y_1)+v_0^*(x_1)+y_0^*(y_1)-x_0^*(x_0,y_0)-v_0^*(x_0)-y_0^*(y_0).$$

Полагая $(x_{m+1},y_{m+1})=(\overline{x},\overline{y})$ и $x_{m+1}^*=x^*$, $v_{m+1}^*=v^*$, $y_{m+1}^*=y^*$ получим

$$g(x,y) \ge x^*_{m+1}(x,y) + v^*_{m+1}(x) + y^*_{m+1}(y) - x^*_{m+1}(x_{m+1},y_{m+1}) - v^*_{m+1}(x_{m+1}) - y^*_{m+1}(y_{m+1}) +$$
$$+ x^*_m(x_{m+1},y_{m+1}) + v^*_m(x_{m+1}) + y^*_m(y_{m+1}) - x^*_m(x_m,y_m) - v^*_m(x_m) - y^*_m(y_m) + x^*_{m-1}(x_m,y_m) + v^*_{m-1}(x_m) +$$
$$+ y^*_{m-1}(y_m) - x^*_{m-1}(x_{m-1},y_{m-1}) - v^*_{m-1}(x_{m-1}) - y^*_{m-1}(y_{m-1}) + \cdots + x^*_1(x_2,y_2) + v^*_1(x_2) + y^*_1(y_2) -$$
$$- x^*_1(x_1,y_1) - v^*_1(x_1) - y^*_1(y_1) + x^*_0(x_1,y_1) + v^*_0(x_1) + y^*_0(y_1) - x^*_0(x_0,y_0) - v^*_0(x_0) - y^*_0(y_0) >$$
$$> x^*(x,y) + v^*(x) + y^*(y) - x^*(\bar{x},\bar{y}) - v^*(\bar{x}) - y^*(\bar{y}) + \alpha$$

в силу определения g. Отсюда следует, что $Df(\bar{x},\bar{y}) \subset Dg(\bar{x},\bar{y})$. Лемма доказана.

3. Характеризация бивыпуклых множеств

Множество $C \subset X \times Y$ называется Bb-выпуклым, если для любого $(\bar{x},\bar{y}) \notin C$ существует $(x^*,v^*,y^*) \in B(X \times Y,R) \times (X \times Y)^*$ такой, что $x^*(\bar{x},\bar{y}) + v^*(\bar{x}) + y^*(\bar{y}) > x^*(x,y) + v^*(x) + y^*(y)$ при $(x,y) \in C$. Отсюда следует, что точка $(\bar{x},\bar{y})$ и множество C «строго биотделимы».

Множество $C \subset X \times Y$ называется сильно Bb-выпуклым, если для любого $(\bar{x},\bar{y}) \notin C$ существуют $(x^*,v^*,y^*) \in B(X \times Y,R) \times (X \times Y)^*$ и $\varepsilon > 0$ такие, что

$$x^*(\bar{x},\bar{y}) + v^*(\bar{x}) + y^*(\bar{y}) - \varepsilon > x^*(x,y) + v^*(x) + y^*(y)$$

при $(x,y) \in C$. Отсюда следует, что точка $(\bar{x},\bar{y})$ и множество C «сильно биотделимы».

Лемма 3.1. Bb –выпуклое множество бивыпукло.

Доказательство. Пусть C Bb-выпуклое множество, $(x_1,y),(x_2,y) \in C$ и существует $\alpha \in (0,1)$, что $(\alpha x_1 + (1-\alpha)x_2, y) \notin C$. Тогда по определению Bb-выпуклого множества существует $(x^*,v^*,y^*) \in B(X \times Y,R) \times (X \times Y)^*$ такой, что

$$x^*(\alpha x_1 + (1-\alpha)x_2, y) + v^*(\alpha x_1 + (1-\alpha)x_2) + y^*(y) > x^*(x_1,y) + v^*(x_1) + y^*(y),$$
$$x^*(\alpha x_1 + (1-\alpha)x_2, y) + v^*(\alpha x_1 + (1-\alpha)x_2) + y^*(y) > x^*(x_2,y) + v^*(x_2) + y^*(y).$$

Отсюда вытекает, что $x^*((1-\alpha)x_2,y) + v^*((1-\alpha)x_2) > x^*((1-\alpha)x_1,y) + v^*((1-\alpha)x_1)$,

$x^*(\alpha x_1,y) + v^*(\alpha x_1) > x^*(\alpha x_2,y) + v^*(\alpha x_2)$, т.е. $x^*(x_2,y) + v^*(x_2) > x^*(x_1,y) + v^*(x_1)$,

$x^*(x_1,y) + v^*(x_1) > x^*(x_2,y) + v^*(x_2)$. Получим противоречие.

Аналогично получим, что если $(x,y_1),(x,y_2) \in C$ и $\beta \in (0,1)$, то $(x, \beta y_1 + (1-\beta)y_2) \in C$. Лемма доказана.

Лемма 3.2. Если $f \in A(X \times Y)$ и $f(x,y) = \sup\limits_{t \in T}(x^*_t(x,y) + v^*_t(x) + y^*_t(y) + c_t)$, где $(x^*_t,v^*_t,y^*_t) \in B(X \times Y,R) \times (X \times Y)^*$, $c_t \in R$ и $C = \{(x,y) \in X \times Y : f(x,y) \le \alpha\} \ne \varnothing$, то C сильно Bb-выпуклое множество.

Доказательство. Если $(\bar{x},\bar{y}) \notin C$, то $f(\bar{x},\bar{y}) > \alpha$. Предположим, что число $f(\bar{x},\bar{y})$ конечно. Положив $\varepsilon = \frac{f(\bar{x},\bar{y}) - \alpha}{3}$ имеем, что $f(\bar{x},\bar{y}) - 2\varepsilon > \alpha$. По условию существует $\tau \in T$ такое, что $f(\bar{x},\bar{y}) - \varepsilon < x^*_\tau(\bar{x},\bar{y}) + v^*_\tau(\bar{x}) + y^*_\tau(\bar{y}) + c_\tau$. Поэтому $f(\bar{x},\bar{y}) - 2\varepsilon < x^*_\tau(\bar{x},\bar{y}) + v^*_\tau(\bar{x}) + y^*_\tau(\bar{y}) + c_\tau - \varepsilon$. Тогда получим, что

$$x^*_t(x,y) + v^*_t(x) + y^*_t(y) + c_t \le f(x,y) \le \alpha < f(\bar{x},\bar{y}) - 2\varepsilon < x^*_\tau(\bar{x},\bar{y}) + v^*_\tau(\bar{x}) + y^*_\tau(\bar{y}) + c_\tau - \varepsilon$$

при $(x,y) \in C$ и $t \in T$. Отсюда следует, что $x^*_\tau(x,y) + v^*_\tau(x) + y^*_\tau(y) < x^*_\tau(\bar{x},\bar{y}) + v^*_\tau(\bar{x}) + y^*_\tau(\bar{y}) - \varepsilon$ при $(x,y) \in C$, т.е. $(x^*_\tau,v^*_\tau,y^*_\tau) \in B(X \times Y,R) \times (X \times Y)^*$ сильно отделяет точку $(\bar{x},\bar{y})$ и множество C.

Если $f(\bar{x},\bar{y}) = +\infty$, то $f(\bar{x},\bar{y}) = \sup\limits_{t \in T}(x^*_t(\bar{x},\bar{y}) + v^*_t(\bar{x}) + y^*_t(\bar{y}) + c_t) = +\infty$. Поэтому существует $\tau \in T$ такое, что $x^*_\tau(\bar{x},\bar{y}) + v^*_\tau(\bar{x}) + y^*_\tau(\bar{y}) + c_\tau - \varepsilon > \alpha$, где $\varepsilon > 0$. Тогда получим, что

$$x^*_\tau(x,y) + v^*_\tau(x) + y^*_\tau(y) < x^*_\tau(\bar{x},\bar{y}) + v^*_\tau(\bar{x}) + y^*_\tau(\bar{y}) - \varepsilon$$

при $(x,y) \in C$, т.е. $(x^*_\tau,v^*_\tau,y^*_\tau) \in B(X \times Y,R) \times (X \times Y)^*$ сильно отделяет точку $(\bar{x},\bar{y})$ и множество C. Лемма доказана.

Следствие 3.1. Если $f \in A(X \times Y)$ и $f(x,y) = \max\limits_{t \in T}(x^*_t(x,y) + v^*_t(x) + y^*_t(y) + c_t)$, где $(x^*_t,v^*_t,y^*_t) \in B(X \times Y,R) \times (X \times Y)^*$, $c_t \in R$ и $C = \{(x,y) \in X \times Y : f(x,y) \le \alpha\} \ne \varnothing$, то C сильно Bb-выпуклое множество.

Если выполнены условия следствия 3.1, то из леммы 3.2 следует, что $C = \{(x,y) \in X \times Y : f(x,y) \le \alpha\}$ сильно Bb -выпуклое множество.

Лемма 3.3. Если $f \in A(X \times Y)$ и $f(x,y) = \sup\limits_{t \in T} (x_t^*(x,y) + v_t^*(x) + y_t^*(y) + c_t)$, где $(x_t^*, v_t^*, y_t^*) \in B(X \times Y, R) \times (X \times Y)^*$, $c_t \in R$ и $2 - ep\ f \ne \varnothing$, то $2 - ep\ f$ сильно Bb -выпуклое множество.

Доказательство. Если $((\overline{x}, \overline{\alpha}), (\overline{y}, \overline{\beta})) \notin 2 - ep\ f$, то $f(\overline{x}, \overline{y}) > \overline{\alpha}\,\overline{\beta}$. Предположим, что число $f(\overline{x}, \overline{y})$ конечно. Поэтому $f(x,y) - \alpha\beta \le 0 < f(\overline{x}, \overline{y}) - \overline{\alpha}\,\overline{\beta}$ при $((x,\alpha),(y,\beta)) \in 2 - ep\ f$. Пусть $0 < 2\varepsilon < f(\overline{x}, \overline{y}) - \overline{\alpha}\,\overline{\beta}$ и $\tau \in T$ такое, что $f(\overline{x}, \overline{y}) - \varepsilon < x_\tau^*(\overline{x}, \overline{y}) + v_\tau^*(\overline{x}) + y_\tau^*(\overline{y}) + c_\tau$. Тогда получим, что

$$x_\tau^*(x,y) + v_\tau^*(x) + y_\tau^*(y) - \alpha\beta + c_\tau \le f(x,y) - \alpha\beta < f(\overline{x}, \overline{y}) - \overline{\alpha}\,\overline{\beta} - 2\varepsilon < x_\tau^*(\overline{x}, \overline{y}) + v_\tau^*(\overline{x}) + y_\tau^*(\overline{y}) + c_\tau - \overline{\alpha}\,\overline{\beta} - \varepsilon$$

при $((x,\alpha),(y,\beta)) \in 2 - ep\ f$. Отсюда следует, что

$$x_\tau^*(x,y) + v_\tau^*(x) + y_\tau^*(y) - \alpha\beta < x_\tau^*(\overline{x}, \overline{y}) + v_\tau^*(\overline{x}) + y_\tau^*(\overline{y}) - \overline{\alpha}\,\overline{\beta} - \varepsilon$$

при $((x,\alpha),(y,\beta)) \in 2 - ep\ f$, т.е. $(x_\tau^*, v_\tau^*, y_\tau^*, -1) \in B(X \times Y, R) \times (X \times Y)^* \times B(R \times R, R)$ сильно отделяет множество $2 - ep\ f$ и точку $((\overline{x}, \overline{\alpha}), (\overline{y}, \overline{\beta}))$. Так как $((\overline{x}, \overline{\alpha}), (\overline{y}, \overline{\beta})) \notin 2 - ep\ f$ любая точка, то $2 - ep\ f$ сильно Bb -выпуклое множество.

Если $f(\overline{x}, \overline{y}) = +\infty$, то $f(\overline{x}, \overline{y}) = \sup\limits_{t \in T} (x_t^*(\overline{x}, \overline{y}) + v_t^*(\overline{x}) + y_t^*(\overline{y}) + c_t) = +\infty$. Поэтому существует $\tau \in T$ такое, что $x_\tau^*(\overline{x}, \overline{y}) + v_\tau^*(\overline{x}) + y_\tau^*(\overline{y}) + c_\tau - \varepsilon > \overline{\alpha}\,\overline{\beta}$, где $\varepsilon > 0$. Тогда получим, что

$$x_\tau^*(x,y) + v_\tau^*(x) + y_\tau^*(y) - \alpha\beta + c_\tau \le f(x,y) - \alpha\beta < x_\tau^*(\overline{x}, \overline{y}) + v_\tau^*(\overline{x}) + y_\tau^*(\overline{y}) + c_\tau - \overline{\alpha}\,\overline{\beta} - \varepsilon$$

при $((x,\alpha),(y,\beta)) \in 2 - ep\ f$. Отсюда следует, что

$$x_\tau^*(x,y) + v_\tau^*(x) + y_\tau^*(y) - \alpha\beta < x_\tau^*(\overline{x}, \overline{y}) + v_\tau^*(\overline{x}) + y_\tau^*(\overline{y}) - \overline{\alpha}\,\overline{\beta} - \varepsilon$$

при $((x,\alpha),(y,\beta)) \in 2 - ep\ f$, т.е. $(x_\tau^*, v_\tau^*, y_\tau^*, -1) \in B(X \times Y, R) \times (X \times Y)^* \times B(R \times R, R)$ сильно отделяет множество $2 - ep\ f$ и точку $((\overline{x}, \overline{\alpha}), (\overline{y}, \overline{\beta}))$.

Так как $((\overline{x}, \overline{\alpha}), (\overline{y}, \overline{\beta})) \notin 2 - ep\ f$ любая точка, то $2 - ep\ f$ сильно Bb -выпуклое множество. Лемма доказана.

Следствие 3.2. Если $f \in A(X \times Y)$ и $f(x,y) = \max\limits_{t \in T} (x_t^*(x,y) + v_t^*(x) + y_t^*(y) + c_t)$, где $(x_t^*, v_t^*, y_t^*) \in B(X \times Y, R) \times (X \times Y)^*$, $c_t \in R$, то $2 - ep\ f$ сильно Bb -выпуклое множество.

Пусть $B_1 \subset X \times Y$ и $B_2 \subset X \times Y$. Множества B_1 и B_2 называются (строго) 2(B)-отделимыми, если существуют $z^* \in (X \times Y)^*$ и $x^* \in B(X \times Y, R)$ такие, что $(x^*, z^*) \ne 0$ и

$$x^*(x_1,y_1) + z^*(x_1,y_1) \ge x^*(x_2,y_2) + z^*(x_2,y_2)$$
$$(x^*(x_1,y_1) + z^*(x_1,y_1) > x^*(x_2,y_2) + z^*(x_2,y_2))$$

при $(x_1,y_1) \in B_1$, $(x_2,y_2) \in B_2$.

Множества B_1 и B_2 называются сильно 2(B)-отделимыми, если существуют $z^* \in (X \times Y)^*$, $x^* \in B(X \times Y, R)$ и $\varepsilon > 0$ такие, что

$$x^*(x_1,y_1) + z^*(x_1,y_1) - \varepsilon \ge x^*(x_2,y_2) + z^*(x_2,y_2)$$

при $(x_1,y_1) \in B_1$, $(x_2,y_2) \in B_2$.

Считаем, что пространство $X \otimes Y \times X \times Y$ снабжено $\sigma(X \otimes Y \times X \times Y, B(X \times Y, R) \times (X \times Y)^*)$ топологией.

Пусть Z линейное пространство, $A \subset Z$. Пересечение всех выпуклых множеств, содержащих множество A, называется выпуклой оболочкой множества A и обозначается через $\operatorname{conv} A$. Отметим, что $\operatorname{conv} A = \{x = \sum\limits_{i=1}^{k} \alpha_i x_i : \sum\limits_{i=1}^{k} \alpha_i = 1,\ \alpha_i \ge 0,\ x_i \in A,\ i = 1,\ldots,k,\ k \in N\}$.

Теорема 3.1. Если $C_1 \subset X \times Y$, $\overline{C}_1 = \{(x_1 \otimes y_1, x_1, y_1) : (x_1,y_1) \in C_1\}$, $C_2 \subset X \times Y$, $\overline{C}_2 = \{(x_2 \otimes y_2, x_2, y_2) : (x_2,y_2) \in C_2\}$, то множества C_1 и C_2 2(B)-отделимы, тогда и только тогда, когда $\operatorname{conv} \overline{C}_1$ и $\operatorname{conv} \overline{C}_2$ отделимы.

Доказательство. Если множества C_1 и C_2 2(B)-отделимы, то существуют $x^* \in B(X \times Y, R)$, $z^* \in (X \times Y)^*$ и число $b \in R$ такие, что $x^*(x_1,y_1) + z^*(x_1,y_1) \le b \le x^*(x_2,y_2) + z^*(x_2,y_2)$ при $(x_1,y_1) \in C_1$

и $(x_2,y_2)\in C_2$, где $(x^*,z^*)\neq 0$. Так как $\overline{C}_1\subset\{(\nu_1,x_1,y_1)\in X\otimes Y\times X\times Y: u(\nu_1)+z^*(x_1,y_1)\le b\}$ и $\overline{C}_2\subset\{(\nu_2,x_2,y_2)\in X\otimes Y\times X\times Y: : u(\nu_2)+z^*(x_2,y_2)\ge b\}$, то имеем, что $\operatorname{conv}\overline{C}_1\subset\{(\nu_1,x_1,y_1)\in X\otimes Y\times X\times Y: u(\nu_1)+z^*(x_1,y_1)\le b\}$ и $\operatorname{conv}\overline{C}_2\subset\{(\nu_2,x_2,y_2)\in X\otimes Y\times X\times Y: u(\nu_2)+z^*(x_2,y_2)\ge b\}$, где $u(\nu)=\sum_{i=1}^{k}x^*(x^i,y^i)$ при $\nu\in X\otimes Y$, $\nu=\sum_{i=1}^{k}x^i\otimes y^i$. Поэтому $u(\nu_1)+z^*(x_1,y_1)\le b$ при $(\nu_1,x_1,y_1)\in\operatorname{conv}\overline{C}_1$, $u(\nu_2)+z^*(x_2,y_2)\ge b$ при $(\nu_2,x_2,y_2)\in\operatorname{conv}\overline{C}_2$. Отсюда следует, что $u(\nu_1)+z^*(x_1,y_1)\le u(\nu_2)+z^*(x_2,y_2)$ при $(\nu_1,x_1,y_1)\in\operatorname{conv}\overline{C}_1$, $(\nu_2,x_2,y_2)\in\operatorname{conv}\overline{C}_2$, т.е. $\operatorname{conv}\overline{C}_1$ и $\operatorname{conv}\overline{C}_2$ отделимы. Обратно, если $\operatorname{conv}\overline{C}_1$ и $\operatorname{conv}\overline{C}_2$ отделимы, то существуют функции $x^*\in B(X\times Y,R)$, $z^*\in(X\times Y)^*$ такие, что $u(\nu_1)+z^*(x_1,y_1)\le u(\nu_2)+z^*(x_2,y_2)$ при $(\nu_1,x_1,y_1)\in\operatorname{conv}\overline{C}_1$, $(\nu_2,x_2,y_2)\in\operatorname{conv}\overline{C}_2$. Поэтому $u(x_1\otimes y_1)+z^*(x_1,y_1)\le u(x_2\otimes y_2)+z^*(x_2,y_2)$ при $(x_1\otimes y_1,x_1,y_1)\in\overline{C}_1$ и $(x_2\otimes y_2,x_2,y_2)\in\overline{C}_2$. Отсюда следует, что $x^*(x_1,y_1)+z^*(x_1,y_1)\le x^*(x_2,y_2)+z^*(x_2,y_2)$ при $(x_1,y_1)\in C_1$ и $(x_2,y_2)\in C_2$, где $x^*(x,y)=u(x\otimes y)$ при $(x,y)\in X\times Y$, т.е. множества C_1 и C_2 2(B)-отделимы. Теорема доказана.

Отметим, что если $C_1\subset X\times Y$, $\overline{C}_1=\{(x_1\otimes y_1,x_1,y_1):(x_1,y_1)\in C_1\}$, $C_2\subset X\times Y$, $\overline{C}_2=\{(x_2\otimes y_2,x_2,y_2):(x_2,y_2)\in C_2\}$, то аналогично проверяется, что множества C_1 и C_2 сильно 2(B)-отделимы (строго 2(B)-отделимы), тогда и только тогда, когда $\operatorname{conv}\overline{C}_1$ и $\operatorname{conv}\overline{C}_2$ сильно отделимы (строго отделимы).

Пусть $C\subset X\times Y$, $\overline{C}=\{(x\otimes y,x,y):(x,y)\in C\}$, $D=(X\times Y)\setminus C=\{(x,y)\in X\times Y:(x,y)\notin C\}$, $\widetilde{D}=\{(x\otimes y,x,y):(x,y)\in D\}$.

Аналогично доказательству теоремы 3.1 получим, что точка $(\overline{x},\overline{y})\notin C$ и множество C сильно 2(B)-отделимы тогда и только тогда, когда $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\notin\overline{\operatorname{conv}}\ \overline{C}$.

Пусть существует $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\notin\overline{C}$ такая, что $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\in\operatorname{conv}\overline{C}$. Тогда существует $(x^i\otimes y^i,x^i,y^i)\in\overline{C}$, $k\in N$ и $\lambda_i>0,\ \sum_{i=1}^{k}\lambda_i=1$ такие, что $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})=\sum_{i=1}^{k}\lambda_i(x^i\otimes y^i,x^i,y^i)$. Отсюда имеем, что $(\overline{x}\otimes\overline{y})=\sum_{i=1}^{k}\lambda_i(x^i\otimes y^i)$, $(\overline{x},\overline{y})=\sum_{i=1}^{k}\lambda_i(x^i,y^i)=(\sum_{i=1}^{k}\lambda_i x^i,\sum_{i=1}^{k}\lambda_i y^i)$. Поэтому $\overline{x}=\sum_{i=1}^{k}\lambda_i x^i$, $\overline{y}=\sum_{i=1}^{k}\lambda_i y^i$.

Тогда получим, что $(\sum_{i=1}^{k}\lambda_i x^i\otimes\sum_{i=1}^{k}\lambda_i y^i)=\sum_{i=1}^{k}\lambda_i(x^i\otimes y^i)$, т.е. $\sum_{i=1}^{k}(\lambda_i^2-\lambda_i)x^i\otimes y^i+\sum_{i=1,j=1,i\neq j}^{k}\lambda_i\lambda_j x^i\otimes y^j=0$. Если $x^i\otimes y^j$ линейно независимы при $i=1,\dots,k$, $j=1,\dots,k$, то существует $s\in\{1,\dots,k\}$ такое, что $\lambda_s=1$ и $\lambda_i=0$ при $i\neq s$. Поэтому $(\overline{x},\overline{y})=(x^s,y^s)$. Так как $(x^s\otimes y^s,x^s,y^s)\in\overline{C}$, то получим, что $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\in\overline{C}$. Получим противоречие.

Лемма 3.4. Если $\overline{\operatorname{conv}}\ \overline{C}\cap\widetilde{D}=\varnothing$ и $(\overline{x},\overline{y})\notin C$, то множество C и точка $(\overline{x},\overline{y})\notin C$ сильно 2(B)-отделимы.

Доказательство. Так как $(\overline{x},\overline{y})\notin C$, то $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\in\widetilde{D}$. Поэтому $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\notin\overline{\operatorname{conv}}\ \overline{C}$. Тогда из теоремы 3.4[7, с.70] следует, что $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})$ и $\overline{\operatorname{conv}}\ \overline{C}$ сильно отделимы. Поэтому точка $(\overline{x},\overline{y})$ и множество C сильно 2(B)-отделимы. Лемма доказана.

Ясно, что если $\overline{\operatorname{conv}}\ \overline{C}\cap\{(x\otimes y,x,y):(x,y)\in X\times Y\}=\overline{C}$, то $\overline{\operatorname{conv}}\ \overline{C}\cap\widetilde{D}=\varnothing$.

Лемма 3.5. Если $(\overline{x},\overline{y})\notin C$, точка $(\overline{x},\overline{y})$ и C строго 2(B)-отделимы, то $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\notin\operatorname{conv}\overline{C}$.

Доказательство. Так как $(\overline{x},\overline{y})\notin C$ и точка $(\overline{x},\overline{y})$ и C строго 2(B)-отделимы, то существуют функции $z^*\in(X\times Y)^*$ и $x^*\in B(X\times Y,R)$ такие, что $(x^*,z^*)\neq 0$ и

$$x^*(\overline{x},\overline{y})+z^*(\overline{x},\overline{y})>x^*(x,y)+z^*(x,y)$$

при $(x,y)\in C$. Покажем, что $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\notin\operatorname{conv}\overline{C}$. Предположим противное. Пусть $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\in\operatorname{conv}\overline{C}$. Тогда существуют $\alpha_i\ge 0,\ \sum_{i=1}^{k}\alpha_i=1$ и $(x^i\otimes y^i,x^i,y^i)\in\overline{C}$ при $i=1,\dots,k$ такие, что

$(\overline{x}\otimes\overline{y},\overline{x},\overline{y})=\sum\limits_{i=1}^{k}\alpha_i(x^i\otimes y^i,x^i,y^i)$. Ясно, что $\widetilde{x}^*(\overline{x},\overline{y})+\widetilde{z}^*(\overline{x},\overline{y})=\sum\limits_{i=1}^{k}\alpha_i(\widetilde{x}^*(x^i,y^i)+\widetilde{z}^*(x^i,y^i))$ при $\widetilde{z}^*\in(X\times Y)^*$ и $\widetilde{x}^*\in B(X\times Y,R)$. По условию имеем, что

$$x^*(\overline{x},\overline{y})+z^*(\overline{x},\overline{y})>x^*(x^i,y^i)+z^*(x^i,y^i)$$

при $i=1,\dots,k$. Поэтому

$$x^*(\overline{x},\overline{y})+z^*(\overline{x},\overline{y})>\sum_{i=1}^{k}\alpha_i(x^*(x^i,y^i)+z^*(x^i,y^i)),$$

т.е. $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\notin\operatorname{conv}\overline{C}$. Получим противоречие. Лемма доказана.

Отметим, что если $(\overline{x},\overline{y})\notin C$, точка $(\overline{x},\overline{y})$ и C сильно 2(B)-отделимы, то аналогично имеем, что $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\notin\overline{\operatorname{conv}}\ \overline{C}$.

Множество $Q\subset X\otimes Y\times X\times Y$ назовем бивыпуклой, если для любого $(x_1\otimes y,x_1,y)$, $(x_2\otimes y,x_2,y)\in Q$, $(x\otimes y_1,x,y_1)$, $(x\otimes y_2,x,y_2)\in Q$ и $\alpha\in[0,1]$ следуют, что $((\alpha x_1+(1-\alpha)x_2)\otimes y,(\alpha x_1+(1-\alpha)x_2),y)\in Q$, $(x\otimes(\alpha y_1+(1-\alpha)y_2),x,(\alpha y_1+(1-\alpha)y_2))\in Q$.

Лемма 3.6. Если $C\subset X\times Y$ бивыпуклое множество, то $\overline{C}=\{(x\otimes y,x,y):(x,y)\in C\}$ бивыпуклое множество.

Доказательство. Если $(x_1\otimes y,x_1,y),(x_2\otimes y,x_2,y)\in\overline{C}=\{(x\otimes y,x,y):(x,y)\in C\}$, то $(x_1,y),(x_2,y)\in C$. Поэтому если $\alpha\in[0,1]$, то $((\alpha x_1+(1-\alpha)x_2),y)\in C$. Тогда получим, что $((\alpha x_1+(1-\alpha)x_2)\otimes y),\alpha x_1+(1-\alpha)x_2,y)\in\overline{C}$. Если $(x\otimes y_1,x,y_1),(x\otimes y_2,x,y_2)\in\overline{C}=\{(x\otimes y,x,y):(x,y)\in C\}$ и $\alpha\in[0,1]$, то аналогично имеем, что $(x\otimes(\alpha y_1+(1-\alpha)y_2),x,\alpha y_1+(1-\alpha)y_2))\in\overline{C}$. Лемма доказана.

Если $\widetilde{C}=\{(\sum\limits_{i=1}^{k}\lambda_i x^i\otimes y^i,x,y):(x^i,y^i)\in C,(x,y)\in C,\lambda_i\ge 0,\sum\limits_{i=1}^{k}\lambda_i=1,k\in N\}$ выпуклое замкнутое множество, то из теоремы 3.4[7 с,70] следует, что множество $C\subset X\times Y$ является сильно Bb -выпуклым. Поэтому множество $C=\{(x,y)\in X\times Y:f(x,y)\le\alpha\}$ является сильно Bb -выпуклым, если $\{(\sum\limits_{i=1}^{k}\lambda_i x^i\otimes y^i,x,y):(x^i,y^i)\in C,(x,y)\in C,\lambda_i\ge 0,\sum\limits_{i=1}^{k}\lambda_i=1,k\in N\}$ выпуклое замкнутое множество.

Отметим, что если $C\subset X\times Y$ выпуклое множество, то

$\widetilde{C}=\{(\sum\limits_{i=1}^{k}\lambda_i x^i\otimes y^i,x,y):(x^i,y^i)\in C,(x,y)\in C,\lambda_i\ge 0,\sum\limits_{i=1}^{k}\lambda_i=1\}$ выпуклое множество.

Рассмотрим конкретные случаи.

1) Пусть $f\in A(X\times Y)$ и $f(x,y)=\sup\limits_{t\in T}(x_t^*(x,y)+z_t^*(x)+y_t^*(y)+c_t)$, $k\in R$ и $C=\{(x,y)\in X\times Y:f(x,y)\le k\}\ne\varnothing$.

Положим

$$C^0=\{(x\otimes y,x,y):(x,y)\in C\},\qquad f^0(x\otimes y,x,y)=\sup_{t\in T}(u_t(x\otimes y)+z_t^*(x)+y_t^*(y)+c_t),$$

где $u_t(x\otimes y)=x_t^*(x,y)$ при $(x,y)\in X\times Y$ имеем, что

$$C^0=\{(x\otimes y,x,y)\in(X\otimes Y)\times(X\times Y):f^0(x\otimes y,x,y)\le k\}.$$

Положим

$$f^{00}(v,x,y)=\sup_{t\in T}(u_t(v)+z_t^*(x)+y_t^*(y)+c_t),$$

где $u_t(v)=\sum\limits_{i=1}^{k}x_t^*(x^i,y^i)$ при $v=\sum\limits_{i=1}^{k}x^i\otimes y^i$, $(x^i,y^i)\in X\times Y$. Из определения $f^{00}(v,x,y)$ следует, что $f^{00}:(X\otimes Y)\times(X\times Y)\to R_{+\infty}$ выпуклая полунепрерывная снизу функция. Поэтому

$$C^{00}=\{(v,x,y)\in(X\otimes Y)\times(X\times Y):f^{00}(v,x,y)\le k\}$$

выпуклое замкнутое множество. Так как $C^0\subset C^{00}$, то имеем, что $\overline{\operatorname{conv}}\,C^0\subset C^{00}$. Если $(\overline{x},\overline{y})\notin C$, то из определения C^0 следует, что $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\notin C^0$, т.е. $f^0(\overline{x}\otimes\overline{y},\overline{x},\overline{y})>k$. Поэтому $f^{00}(\overline{x}\otimes\overline{y},\overline{x},\overline{y})=f^0(\overline{x}\otimes\overline{y},\overline{x},\overline{y})>k$. Отсюда следует, что $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})\notin C^{00}$. Тогда из теоремы 3.4[7] следует, что $(\overline{x}\otimes\overline{y},\overline{x},\overline{y})$ и C^{00} сильно отделимы. Поэтому точка $(\overline{x},\overline{y})$ и множество C сильно 2(B)-отделимы.

2) Пусть $f \in A(X \times Y)$, где $f(x,y) = \sup\limits_{t \in T} (x_t^*(x,y) + v_t^*(x) + y_t^*(y) + c_t)$ и

$2 - E(f) = C = \{(x,y,\alpha,\beta) \in X \times Y \times R \times R : f(x,y) \le \alpha\beta\} \neq \varnothing$. Положим

$$C^0 = \{(x \otimes y, x, y, \lambda) : (x,y,\alpha\beta) \in C, \ \lambda = \alpha\beta\},$$

$$f^0(x \otimes y, x, y) = \sup_{t \in T} (u_t(x \otimes y) + v_t^*(x) + y_t^*(y) + c_t),$$

где $u_t(x \otimes y) = x_t^*(x,y)$ при $(x,y) \in X \times Y$ имеем, что

$$C^0 = \{(x \otimes y, x, y, \alpha) \in (X \otimes Y) \times (X \times Y) \times R : f^0(x \otimes y, x, y) \le \alpha\}.$$

Положим

$$f^{00}(\nu, x, y) = \sup_{t \in T} (u_t(\nu) + v_t^*(x) + y_t^*(y) + c_t),$$

где $u_t(\nu) = \sum\limits_{i=1}^{k} x_t^*(x^i, y^i)$ при $\nu = \sum\limits_{i=1}^{k} x^i \otimes y^i$, $(x^i, y^i) \in X \times Y$. Из определения $f^{00}(\nu, x, y)$ следует, что $f^{00} : (X \otimes Y) \times (X \times Y) \to R_{+\infty}$ выпуклая полунепрерывная снизу функция. Поэтому

$$C^{00} = \{(\nu, x, y, \alpha) \in (X \otimes Y) \times (X \times Y) \times R : f^{00}(\nu, x, y) \le \alpha\}$$

выпуклое замкнутое множество. Так как $C^0 \subset C^{00}$, то имеем, что $\overline{\text{conv}}\, C^0 \subset C^{00}$. Если $(\overline{x}, \overline{y}, \overline{\alpha}, \overline{\beta}) \notin C$, то из определения C^0 следует, что $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\lambda}) \notin C^0$, где $\overline{\lambda} = \overline{\alpha}\,\overline{\beta}$, т.е. $f^0(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}) > \overline{\lambda}$. Поэтому $f^{00}(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}) = f^0(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}) > \overline{\lambda}$. Отсюда следует, что $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\lambda}) \notin C^{00}$. Тогда из теоремы 3.4[7, с.70] следует, что $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\lambda})$ и C^{00} сильно отделимы. Поэтому точка $(\overline{x}, \overline{y}, \overline{\alpha}, \overline{\beta})$ и множество C сильно 2(B)-отделимы.

Отметим, что если $f_0 : X \times Y \to R_{+\infty}$ каноническая бивыпуклая функция, $f_1 : X \times Y \to R_{+\infty}$ собственная полунепрерывная снизу выпуклая функция, то существуют множества индексов T и S такие, что

$$f(x,y) = f_0(x,y) + f_1(x,y) = \sup_{t \in T} (x_t^*(x,y) + c_t) +$$

$$+ \sup_{s \in S} (v_s^*(x) + y_s^*(y) + d_s) = \sup_{t \in T, s \in S} (x_t^*(x,y) + c_t + v_s^*(x) + y_s^*(y) + d_s)$$

при $(x,y) \in X \times Y$. Поэтому, если

$$2 - E(f) = C = \{(x,y,\alpha,\beta) \in X \times Y \times R \times R : f(x,y) \le \alpha\beta\} \neq \varnothing$$

и $(\overline{x}, \overline{y}, \overline{\alpha}, \overline{\beta}) \notin C$, то точка $(\overline{x}, \overline{y}, \overline{\alpha}, \overline{\beta})$ и множество C сильно 2(B)-отделимы.

Пусть $f : X \times Y \to R$ бивыпуклая функция, $C = \{(x,y,\lambda) \in X \times Y \times R : f(x,y) \le \lambda\}$. Положим $C^0 = \{(x \otimes y, x, y, \lambda) : (x,y,\lambda) \in C\}$. Если из $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\lambda}) \notin C^0$ следует, что $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\lambda}) \notin \overline{\text{conv}}\, C^0$, то из теоремы 3.4[7] следует, что $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\lambda})$ и C^0 сильно отделимы.

Лемма 3.7. Если $y^1, y^2, ..., y^k \in Y$ линейно независимы и $\sum\limits_{i=1}^{k} x^i \otimes y^i = x \otimes y$, то пары $\{x^i, x\}$ линейно зависимы при $i = 1, \ldots, k$, т.е. существуют числа β_i такие, что $x^i = \beta_i x$ при $i = 1, \ldots, k$.

Доказательство. Предположим противное. Пусть $x \neq 0$ и для некоторого $j \in \{1, \ldots, k\}$ пары $\{x_1^j, x\}$ линейно независимы. Тогда по лемме о биортогональном базисе (см. [2, с.25]) существует функция $x^* \in X^*$ такая, что $x^*(x) = 0$ и $x^*(x^j) = 1$. Также по лемме о биортогональном базисе существует функция $y^* \in Y^*$ такая, что $y^*(y^s) = 0$ при $s \neq j$ и $y^*(y^j) = 1$. Ясно, что $(x^* \otimes y^*)(x^j, y^j) = x^*(x^j) y^*(y^j) = 1$. Так как $\sum\limits_{i=1}^{k} x^*(x^i) y^*(y^i) = 1$ и $x^*(x) y^*(y) = 0$, то получим противоречие, т.е. пары $\{x^i, x\}$ линейно зависимы при $i = 1, \ldots, k$. Поэтому существуют числа β_i такие, что $x^i = \beta_i x$ при $i = 1, \ldots, k$.

Ясно, что если $x^j = 0$, то $\beta_j = 0$.

Если $x = 0$ или $y = 0$, то из равенства $\sum\limits_{i=1}^{k} x^i \otimes y^i = x \otimes y$ следует, что $x^i = 0$ при $i = 1, \ldots, k$. Поэтому пары $\{x^i, x\}$ линейно зависимы при $i = 1, \ldots, k$. Лемма доказана.

Леммы 3.8. Если $f : X \times Y \to R$ бивыпуклая функция и

$$C^0 = \{(x \otimes y, x, y, \alpha) \in (X \otimes Y) \times (X \times Y) \times R : f(x,y) \le \alpha\},$$

$$\widetilde{C}^{00} = \{(\nu, x, y, \alpha) \in X \otimes Y \times X \times Y \times R : (\nu, x, y, \alpha) = \sum_{i=1}^{k} \lambda_i (x^i \otimes y^i, x^i, y^i, \alpha_i),$$

где $(x^i \otimes y^i, x^i, y^i, \alpha_i) \in C^0$, $\lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1$, $k \in N$, $y^1, y^2, ..., y^k \in Y$ линейно независимы$\}$, то из соотношения $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\alpha}) \notin C^0$ следует, что $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\alpha}) \notin \widetilde{C}^{00}$.

Доказательство. Предположим противное. Пусть существует точка $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\alpha}) \notin C^0$ такая, что $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\alpha}) \in \widetilde{C}^{00}$. Тогда существует $(x^i \otimes y^i, x^i, y^i, \alpha_i) \in C^0$, $k \in N$ и $\lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1$ такие, что $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\alpha}) = \sum_{i=1}^{k} \lambda_i (x^i \otimes y^i, x^i, y^i, \alpha_i)$. Отсюда имеем, что $\overline{x} \otimes \overline{y} = \sum_{i=1}^{k} \lambda_i (x^i \otimes y^i)$, $(\overline{x}, \overline{y}) = \sum_{i=1}^{k} \lambda_i (x^i, y^i)$ и $\overline{\alpha} = \sum_{i=1}^{k} \lambda_i \alpha_i$.

Пусть $\overline{x} \neq 0$. Так как $y^1, y^2, ..., y^k \in Y$ линейно независимы и $\overline{x} \otimes \overline{y} = \sum_{i=1}^{k} \lambda_i x^i \otimes y^i$, где $\lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1$, то из леммы 3.7 следует, что пары $\{\lambda_i x^i, \overline{x}\}$ линейно зависимы при $i = 1, \ldots, k$, т.е. существуют числа β_i такие, что $\lambda_i x^i = \beta_i \overline{x}$ при $i = 1, \ldots, k$. Поэтому $\overline{x} \otimes \overline{y} = \sum_{i=1}^{k} \beta_i \overline{x} \otimes y^i$. Тогда получим, что $\overline{x} \otimes \overline{y} = \sum_{i=1}^{k} (\overline{x} \otimes \beta_i y^i) = (\overline{x} \otimes \sum_{i=1}^{k} \beta_i y^i)$. Отсюда следует, что $\overline{y} = \sum_{i=1}^{k} \beta_i y^i$. Из равенства $(\overline{x}, \overline{y}) = \sum_{i=1}^{k} \lambda_i (x^i, y^i)$ следует, что $\overline{y} = \sum_{i=1}^{k} \lambda_i y^i$. Так как $y^1, y^2, ..., y^k \in Y$ линейно независимы, то из равенства $\sum_{i=1}^{k} \beta_i y^i = \sum_{i=1}^{k} \lambda_i y^i$ следует, что $\sum_{i=1}^{k} (\beta_i - \lambda_i) y^i = 0$. Поэтому $\beta_i - \lambda_i = 0$. Отсюда следует, что $\beta_i = \lambda_i$ при $i = 1, \ldots, k$ и $x^i = \overline{x}$. Тогда имеем, что

$$\overline{\alpha} = \sum_{i=1}^{k} \lambda_i \alpha_i \geq \sum_{i=1}^{k} \lambda_i f(x^i, y^i) = \sum_{i=1}^{k} \lambda_i f(\overline{x}, y^i) \geq f(\overline{x}, \sum_{i=1}^{k} \lambda_i y^i) = f(\overline{x}, \overline{y}) > \overline{\alpha}.$$

Получим противоречие.

Если $x^j = 0$ при некоторой $1 \leq j \leq k$, то $\beta_j = 0$. Так как $\lambda_j \neq 0$, то из равенства $\sum_{i=1}^{k} (\beta_i - \lambda_i) y^i = 0$ следует, что $\beta_j = \lambda_j$. Получим противоречие.

Если $\overline{x} = 0$, то из равенства $\overline{x} \otimes \overline{y} = \sum_{i=1}^{k} \lambda_i (x^i \otimes y^i)$ следует, что $x^i = 0$ при $i = 1, \ldots, k$. Так как $\overline{y} = \sum_{i=1}^{k} \lambda_i y^i$, то $\overline{\alpha} = \sum_{i=1}^{k} \lambda_i \alpha_i \geq \sum_{i=1}^{k} \lambda_i f(0, y^i) \geq f(0, \sum_{i=1}^{k} \lambda_i y^i) = f(0, \overline{y}) > \overline{\alpha}$. Получим противоречие. Лемма доказана.

Легко проверяется, что если $y^1, y^2, ..., y^k \in Y$ линейно независимы и $x^i \neq 0$ при $i = 1, \ldots, k$, то $x^i \otimes y^i$ линейно независимы при $i = 1, \ldots, k$.

Лемма 3.9. Если $f \in A(X \times Y)$, т.е. существует множество индексов T такое, что $f(x, y) = \sup_{t \in T} (x_t^*(x, y) + v_t^*(x) + y_t^*(y) + c_t)$ при $(x, y) \in X \times Y$, где $x_t^* \in B(X \times Y, R)$, $v_t^* \in X^*, y_t^* \in Y^*$ и $c_t \in R$,

$$C^0 = \{(x \otimes y, x, y, \alpha) \in (X \otimes Y) \times (X \times Y) \times R : f(x, y) \leq \alpha\},$$

$$C^{00} = \{(\nu, x, y, \alpha) \in X \otimes Y \times X \times Y \times R : (\nu, x, y, \alpha) =$$

$$= \sum_{i=1}^{k} \lambda_i (x^i \otimes y^i, x^i, y^i, \alpha_i), \text{ где } (x^i \otimes y^i, x^i, y^i, \alpha_i) \in C^0, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1, k \in N\},$$

то из соотношения $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\alpha}) \notin C^0$ следует, что $(\overline{x} \otimes \overline{y}, \overline{x}, \overline{y}, \overline{\alpha}) \notin C^{00}$.

Доказательство. Предположим противное. Пусть существует точка $(\overline{x}\otimes\overline{y},\overline{x},\overline{y},\overline{\alpha})\notin C^0$ такая, что $(\overline{x}\otimes\overline{y},\overline{x},\overline{y},\overline{\alpha})\in C^{00}$. Тогда существует $(x^i\otimes y^i,x^i,y^i,\alpha_i)\in C^0,\quad k\in N$ и $\lambda_i\geq 0,\sum\limits_{i=1}^{k}\lambda_i=1$ такие, что $(\overline{x}\otimes\overline{y},\overline{x},\overline{y},\overline{\alpha})=\sum\limits_{i=1}^{k}\lambda_i(x^i\otimes y^i,x^i,y^i,\alpha_i)$. Отсюда имеем, что $\overline{x}\otimes\overline{y}=\sum\limits_{i=1}^{k}\lambda_i(x^i\otimes y^i)$,

$(\overline{x},\overline{y})=\sum\limits_{i=1}^{k}\lambda_i(x^i,y^i)$ и $\overline{\alpha}=\sum\limits_{i=1}^{k}\lambda_i\alpha_i$. Так как $f(x,y)=\sup\limits_{t\in T}(x_t^*(x,y)+v_t^*(x_1)+y_t^*(y)+c_t)$, то

$$\overline{\alpha}=\sum_{i=1}^{k}\lambda_i\alpha_i\geq\sum_{i=1}^{k}\lambda_i f(x^i,y^i)=\sum_{i=1}^{k}\lambda_i\sup_{t\in T}(x_t^*(x^i,y^i)+v_t^*(x^i)+y_t^*(y^i)+c_t)\geq$$

$$\geq\sup_{t\in T}\sum_{i=1}^{k}\lambda_i(x_t^*(x^i,y^i)+v_t^*(x^i)+y_t^*(y^i)+c_t)=\sup_{t\in T}(x_t^*(\sum_{i=1}^{k}\lambda_i x^i\otimes y^i)+$$

$$+v_t^*(\sum_{i=1}^{k}\lambda_i x^i)+y_t^*(\sum_{i=1}^{k}\lambda_i y^i)+c_t)=\sup_{t\in T}(x_t^*(\overline{x}\otimes\overline{y})+v_t^*(\overline{x})+y_t^*(\overline{y})+c_t)=f(\overline{x},\overline{y})>\overline{\alpha}.$$

Получим противоречие. Лемма доказана.

Если $q_0:X\times Y\to R$ неотрицательная четная бисублинейная функция, $q_1:X\times Y\to R$ неотрицательная сублинейная функция, то положим $\overline{q}_0(x\otimes y)=q_0(x,y)$ и

$$\overline{C}=\{(x\otimes y,x,y)\in X\otimes Y\times X\times Y:\overline{q}_0(x\otimes y)+q_1(x,y)\leq 1\}.$$

Положив

$$p_C(x,y)=p_{\overline{C}}(x\otimes y,x,y)=\inf\left\{\alpha:\frac{(x\otimes y,x,y)}{\alpha}\in\overline{C},\ \ \alpha>0\right\},$$

рассмотрим аналог функции Минковского, где $(x,y)\in X\times Y$. Тогда получим, что

$$p_C(x,y)=\inf\{\alpha:\overline{q}_0(\tfrac{1}{\alpha}x\otimes y)+q_1(\frac{x,y}{\alpha})\leq 1,\ \alpha>0\}=$$

$$=\inf\{\alpha:\overline{q}_0(x\otimes y)+q_1(x,y)\leq\alpha,\ \ \alpha>0\}=q_0(x,y)+q_1(x,y)$$

при $(x,y)\in X\times Y$.

Если $f\in SA(X\times Y)$, существует множество индексов T такое, что

$$f(x,y)=\sup_{t\in T}(x_t^*(x,y)+v_t^*(x)+y_t^*(y))$$

и $f(x,y)\geq 0$ при $(x,y)\in X\times Y$ и $\overline{C}=\{(x\otimes y,x,y)\in X\otimes Y\times X\times Y:\sup\limits_{t\in T}(x_t^*(x\otimes y)+v_t^*(x)+y_t^*(y))\leq 1\}$, то

$$p_C(x,y)=p_{\overline{C}}(x\otimes y,x,y)=\inf\{\alpha:\sup_{t\in T}(x_t^*(\tfrac{1}{\alpha}x\otimes y)+v_t^*(\tfrac{1}{\alpha}x)+y_t^*(\tfrac{1}{\alpha}y))\leq 1,\ \alpha>0\}=$$

$$=\inf\{\alpha:\ \sup_{t\in T}(x_t^*(x\otimes y)+v_t^*(x)+y_t^*(y))\leq\alpha,\ \ \alpha>0\}=\sup_{t\in T}(x_t^*(x,y)+v_t^*(x)+y_t^*(y))$$

при $(x,y)\in X\times Y$.

Пусть $f:X\times Y\to R_{+\infty}$. Положим

$$2-E(f)=\{(x,y,\alpha,\beta)\in X\times Y\times R\times R:f(x,y)\leq\alpha\beta\}.$$

Если $f(x,y)=f^{**}(x,y)$ при $(x,y)\in X\times Y$, то

$$f(x,y)=\sup_{(x^*,v^*,y^*)\in\operatorname{dom}f^*}\{x^*(x,y)+v^*(x)+y^*(y)-f^*(x^*,v^*,y^*)\}$$

при $(x,y)\in X\times Y$. Кроме того, если $f:X\times Y\to R$, то из леммы 3.3 следует, что $2-epf$ сильно Bb -выпуклое множество. Докажем обратное утверждение.

Теорема 3.2. Если $f:X\times Y\to R$ такая функция, что $2-E(f)$ строго Bb -выпуклое множество, то $f(x,y)=f^{**}(x,y)$ при $(x,y)\in X\times Y$.

Доказательство. Пусть $(\overline{x},\overline{y},\overline{\alpha},\overline{\beta})\notin 2-E(f)$, т.е. $f(\overline{x},\overline{y})>\overline{\alpha}\overline{\beta}$. Так как $2-E(f)$ строго

Bb -выпуклое множество, то существуют $x^*\in B(X\times Y,R)$, $(v^*,y^*)\in(X\times Y)^*$ и $b\in R$, что $x^*(x,y)+v^*(x)+y^*(y)-b\alpha\beta<x^*(\overline{x},\overline{y})+v^*(\overline{x})+y^*(\overline{y})-b\overline{\alpha}\overline{\beta}$ при $(x,y,\alpha,\beta)\in 2-E(f)$. Положив $\alpha\beta=f(\overline{x},\overline{y})$ получим, что $x^*(\overline{x},\overline{y})+v^*(\overline{x})+y^*(\overline{y})-bf(\overline{x},\overline{y})<x^*(\overline{x},\overline{y})+v^*(\overline{x})+y^*(\overline{y})-b\overline{\alpha}\overline{\beta}$, т.е. $b(f(\overline{x},\overline{y})-\overline{\alpha}\overline{\beta})>0$. Отсюда вытекает, что $b>0$. Тогда имеем, что

$\frac{1}{b}x^*(x,y)+\frac{1}{b}v^*(x)+\frac{1}{b}y^*(y)-\alpha\beta<\frac{1}{b}x^*(\bar{x},\bar{y})+\frac{1}{b}v^*(\bar{x})+\frac{1}{b}y^*(\bar{y})-\bar{\alpha}\bar{\beta}$ при $(x,y,\alpha,\beta)\in 2-E(f)$. Возьмем $\alpha\beta=f(x,y)$, $\bar{x}^*(x,y)=\frac{1}{b}x^*(x,y)$, $\bar{v}^*(x)=\frac{1}{b}v^*(x)$, $\bar{y}^*(y)=\frac{1}{b}y^*(y)$. Ясно, что

$\bar{x}^*(x,y)+\bar{v}^*(x)+\bar{y}^*(y)-f(x,y)<\bar{x}^*(\bar{x},\bar{y})+\bar{v}^*(\bar{x})+\bar{y}^*(\bar{y})-\bar{\alpha}\bar{\beta}$ при $(x,y)\in X\times Y$. Поэтому

$$\sup_{(x,y)\in X\times Y}\{\bar{x}^*(x,y)+\bar{v}^*(x)+\bar{y}^*(y)-f(x,y)\}\le\bar{x}^*(\bar{x},\bar{y})+\bar{v}^*(\bar{x})+\bar{y}^*(\bar{y})-\bar{\alpha}\bar{\beta},$$

т.е. $f^*(\bar{x}^*,\bar{v}^*,\bar{y}^*)\le\bar{x}^*(\bar{x},\bar{y})+\bar{v}^*(\bar{x})+\bar{y}^*(\bar{y})-\bar{\alpha}\bar{\beta}$. Отсюда имеем, что $(\bar{x}^*,\bar{v}^*,\bar{y}^*)\in\operatorname{dom}f^*$ и $\bar{\alpha}\bar{\beta}\le\bar{x}^*(\bar{x},\bar{y})+\bar{v}^*(\bar{x})+\bar{y}^*(\bar{y})-f^*(\bar{x}^*,\bar{v}^*,\bar{y}^*)\le f^{**}(\bar{x},\bar{y})$. Получим, что $\bar{\alpha}\bar{\beta}\le f^{**}(\bar{x},\bar{y})\le f(\bar{x},\bar{y})$. Поэтому $f^{**}(\bar{x},\bar{y})=f(\bar{x},\bar{y})$. Теорема доказана.

Теорема 3.3. Если $f:X\times Y\to R_{+\infty}$ собственная функция и $2-E(f)$ сильно Bb -выпуклое множество, то $f(x,y)=f^{**}(x,y)$ при $(x,y)\in X\times Y$.

Доказательство. Пусть $(\bar{x},\bar{y},\bar{\alpha},\bar{\beta})\notin 2-E(f)$, т.е. $f(\bar{x},\bar{y})>\bar{\alpha}\bar{\beta}$. Так как $2-E(f)$ сильно Bb -выпуклое множество, то существуют $x^*\in B(X\times Y,R)$, $(v^*,y^*)\in(X\times Y)^*$ и $b\in R$, что $x^*(x,y)+v^*(x)+y^*(y)-b\alpha\beta<x^*(\bar{x},\bar{y})+v^*(\bar{x})+y^*(\bar{y})-b\bar{\alpha}\bar{\beta}-\varepsilon$ при $(x,y,\alpha,\beta)\in 2-E(f)$. Положив $(x,y)=(\tilde{x},\tilde{y})\in\operatorname{dom}f$ и $f(\tilde{x},\tilde{y})\le\alpha\beta$ получим, что

$$x^*(\tilde{x},\tilde{y})+v^*(\tilde{x})+y^*(\tilde{y})-b\alpha\beta<x^*(\bar{x},\bar{y})+v^*(\bar{x})+y^*(\bar{y})-b\bar{\alpha}\bar{\beta}-\varepsilon.$$

Отсюда при $\alpha\beta\to+\infty$ имеем, что $b\ge 0$.

Если $b>0$, то аналогично доказательству теоремы 3.2 имеем, что $\frac{1}{b}(x^*,v^*,y^*)\in\operatorname{dom}f^*$ и $f^{**}(\bar{x},\bar{y})=f(\bar{x},\bar{y})$. Также аналогично доказательству теоремы 3.2 имеем, что $b>0$ при $(\bar{x},\bar{y})\in\operatorname{dom}f$.

Пусть $(\bar{x},\bar{y})\notin\operatorname{dom}f$. Тогда, если $b>0$, то имеем, что $f^{**}(\bar{x},\bar{y})=f(\bar{x},\bar{y})$. Если $b=0$, то $x^*(x,y)+v^*(x)+y^*(y)\le x^*(\bar{x},\bar{y})+v^*(\bar{x})+y^*(\bar{y})-\varepsilon$ при $(x,y)\in X\times Y$. Возьмем $(\bar{x}^*,\bar{v}^*,\bar{y}^*)\in\operatorname{dom}f^*$. Тогда

$f^*(\bar{x}^*,\bar{v}^*,\bar{y}^*)\ge\bar{x}^*(x,y)+\bar{v}^*(x)+\bar{y}^*(y)-f(x,y)$ при $(x,y)\in\operatorname{dom}f$. Поэтому

$$\bar{x}^*(x,y)+\bar{v}^*(x)+\bar{y}^*(y)+\mu x^*(x,y)+\mu(v^*(x)+y^*(y))-f(x,y)\le$$
$$\le f^*(\bar{x}^*,\bar{v}^*,\bar{y}^*)+\mu x^*(\bar{x},\bar{y})+\mu(v^*(\bar{x})+y^*(\bar{y}))-\mu\varepsilon$$

при $\mu>0$. Отсюда следует

$$f^*(\bar{x}^*+\mu x^*,\bar{v}^*+\mu v^*,\bar{y}^*+\mu y^*)\le f^*(\bar{x}^*,\bar{v}^*,\bar{y}^*)+\mu x^*(\bar{x},\bar{y})+\mu(v^*(\bar{x})+y^*(\bar{y}))-\mu\varepsilon,$$

которую можно записать в виде

$$\bar{x}^*(\bar{x},\bar{y})+\bar{v}^*(\bar{x})+\bar{y}^*(\bar{y})+\mu\varepsilon-f^*(\bar{x}^*,\bar{v}^*,\bar{y}^*)\le(\bar{x}^*+\mu x^*)(\bar{x},\bar{y})+$$
$$+(\bar{v}^*+\mu v^*)(\bar{x})+(\bar{y}^*+\mu y^*)(\bar{y})-f^*(\bar{x}^*+\mu x^*,\bar{v}^*+\mu v^*,\bar{y}^*+\mu y^*)\le f^{**}(\bar{x},\bar{y})$$

Подставив $\mu=\frac{a+f^*(\bar{x}^*,\bar{v}^*,\bar{y}^*)-\bar{x}^*(\bar{x},\bar{y})-\bar{v}^*(\bar{x})-\bar{y}^*(\bar{y})}{\varepsilon}$ отсюда имеем, что $a\le f^{**}(\bar{x},\bar{y})$ при достаточно больших a. Поэтому $f^{**}(\bar{x},\bar{y})=+\infty$. Теорема доказана.

Если $2-E(f)$ сильно Bb -выпуклое множество, то $f(x,y)=f^{**}(x,y)$ при $(x,y)\in X\times Y$. Поэтому $f(x,y)=\sup_{(x^*,v^*,y^*)\in\operatorname{dom}f^*}\{x^*(x,y)+v^*(x)+y^*(y)-f^*(x^*,v^*,y^*)\}$ при $(x,y)\in X\times Y$, т.е. $f\in A(X\times Y)$. Если $f\in A(X\times Y)$ и $(\bar{x},\bar{y},\bar{\alpha},\bar{\beta})\notin 2-E(f)$, то из леммы 3.3(или из 2)) следует, что точка $(\bar{x},\bar{y},\bar{\alpha},\bar{\beta})$ и множество $2-E(f)$ сильно 2(B)-отделимы. Тогда из теоремы 3.3 следует следующее следствие.

Следствие 3.3. $2-E(f)$ сильно Bb -выпуклое множество в том и только в том случае, когда $f\in A(X\times Y)$.

Если $2-E(f)$ сильно Bb -выпуклое множество, то функцию $f:X\times Y\to R_{+\infty}$ назовем (A)-выпуклой.

Другие варианты теоремы 3.2 и 3.3 доказаны в [11, с. 96].

§4. Исследование свойств 2-сублинейных функций

Пусть X и Y действительные банаховы пространства, $x^*\in B(X\times Y,R)$, $(v^*,y^*)\in(X\times Y)^*$. Функцию $b(x,y)=x^*(x,y)+v^*(x)+y^*(y)$ назовем 2-линейной непрерывной из $X\times Y$ в R.

Если $f: X\times Y\to R_{+\infty}$ бивыпуклая функция, функции $f(x,0)$ и $f(0,y)$ сублинейные на X и Y соответственно и $f(0,0)=0$, то функцию f назовем 2–сублинейной.

Совокупность собственных функций $f: X\times Y\to R_{+\infty}$, каждая из которых является поточечной верхней гранью некоторого семейства 2-линейных непрерывных функций обозначим через $SA(X\times Y)$. Функцию $q\in SA(X\times Y)$ назовем канонической 2–сублинейной.

Если $f_1: X\times Y\to R_{+\infty}$ бисублинейная функция и $f_2: X\times Y\to R_{+\infty}$ сублинейная функция, то $f_1+f_2: X\times Y\to R_{+\infty}$ 2-сублинейная функция. Поэтому если $A_1\subset B(X\times Y,R)$ и $A_2\subset X^*\times Y^*$ непустые множества, то $f(x,y)=\sup\limits_{x^*\in A_1} x^*(x,y)+\sup\limits_{(v^*,y^*)\in A_2}(v^*(x)+y^*(y))$ 2-сублинейная функция.

Отметим, что бисублинейная и каноническая 2–сублинейная функция является 2–сублинейной функцией.

Если $F: X\otimes Y\times X\times Y\to R_{+\infty}$ выпуклая положительно однородная функция и $F(0,0,0)=0$, то функция называется сублинейной. Множество всех собственных полунепрерывных снизу сублинейных функций из $X\otimes Y\times X\times Y$ в $R_{+\infty}$ обозначим через $S(X\otimes Y\times X\times Y)$.

Если $f\in SA(X\times Y)$, то существует множество индексов T такое, что

$$f(x,y)=\sup_{t\in T}(x_t^*(x,y)+v_t^*(x)+y_t^*(y))$$

при $(x,y)\in X\times Y$, где $x_t^*\in B(X\times Y,R)$, $v_t^*\in X^*$, $y_t^*\in Y^*$. Ясно, что

$$g(\nu,x,y)=\sup_{t\in T}(\sum_{i=1}^{k}x_t^*(x^i,y^i)+v_t^*(x)+y_t^*(y)),$$

где $\nu=\sum\limits_{i=1}^{k}x^i\otimes y^i$, принадлежит $S(X\otimes Y\times X\times Y)$.

Отметим, что если $f\in SA(X\times Y)$, то существует множество $S\subset B(X\times Y,R)\times(X\times Y)^*$ такое, что $f(x,y)=\sup\limits_{(x^*,z^*,y^*)\in S}(x^*(x,y)+v^*(x)+y^*(y))$ при $(x,y)\in X\times Y$.

Обратно, если $F\in S(X\otimes Y\times X\times Y)$, то из предложения 4.1.1[2, с.203] следует, что функция $F(\nu,x,y)$ является поточечной верхней гранью некоторого непустого семейства линейных функций вида $b(\nu,x,y)=\sum\limits_{i=1}^{k}x^*(x^i,y^i)+v^*(x)+y^*(y)$, где $x^*\in B(X\times Y,R)$, $v^*\in X^*$, $y^*\in Y^*$ и $\nu=\sum\limits_{i=1}^{k}x^i\otimes y^i$. Поэтому существует множество индексов T такое, что

$$F(\nu,x,y)=\sup_{t\in T}b_t(\nu,x,y)=\sup_{t\in T}(\sum_{i=1}^{k}x_t^*(x^i,y^i)+v_t^*(x)+y_t^*(y)).$$

Отсюда следует, что функция $\varphi(x,y)=F(x\otimes y,x,y)=\sup\limits_{t\in T}(x_t^*(x,y)+z_t^*(x)+y_t^*(y))$, где $(x,y)\in X\times Y$, принадлежит в $SA(X\times Y)$, т.е. $\varphi\in SA(X\times Y)$. Поэтому получим, что

$$SA(X\times Y)=\{\varphi(x,y)=f(x\otimes y,x,y)\ \text{при}\ (x,y)\in X\times Y: f\in S(X\otimes Y\times X\times Y)\}.$$

Отметим, что если $F(\nu,x,y)$ является поточечной верхней гранью некоторого непустого семейства линейных непрерывных функций вида $b(\nu,x,y)$, то из предложения 4.1.1[2, с.203] и предложения 1.3.1[17, с.24] следует, что $F(\nu,x,y)$ выпуклая полунепрерывная снизу положительно однородная функция в $X\otimes Y\times X\times Y$.

Следующие утверждения доказываются аналогично утверждениям в пункте 2.

Лемма 4.1. Если $F: X\otimes Y\times X\times Y\to R_{+\infty}$ сублинейная функция, то $\varphi(x,y)=F(x\otimes y,x,y)$ 2-сублинейная функция на $X\times Y$.

Отсюда следует, что если $F\in S(X\otimes Y\times X\times Y)$, то $\varphi(x,y)=F(x\otimes y,x,y)$ при $(x,y)\in X\times Y$ также 2-сублинейная функция на $X\times Y$.

Лемма 4.2. Если $\varphi\in SA(X\times Y)$ и $\varphi(x,y)=\sup\limits_{t\in T}(x_t^*(x,y)+v_t^*(x)+y_t^*(y))$ и существуют числа L_0, L_1, L_2 такие, что $|x_t^*(x,y)|\le L_0\|x\|\|y\|$, $|z_t^*(x)|\le L_1\|x\|$, $|y_t^*(y)|\le L_2\|y\|$ при $(x,y)\in X\times Y$ и $t\in T$, то $\varphi(x,y)$ непрерывная функция.

Пусть $f: X\times Y\to R_{+\infty}$ и $(\overline{x},\overline{y})\in\operatorname{dom} f$. Положим

$$D f(\overline{x},\overline{y})=\{(x^*,v^*,y^*)\in B(X\times Y,R)\times X^*\times Y^*: f(x,y)-f(\overline{x},\overline{y})\ge$$
$$\ge x^*(x,y)+v^*(x)+y^*(y)-x^*(\overline{x},\overline{y})-x^*(\overline{x})-y^*(\overline{y})\ \text{при}\ (x,y)\in X\times Y\}.$$

Лемма 4.3. Если $\varphi \in SA(X\times Y)$ и $\varphi(x,y)=\sup\limits_{t\in T}(x_t^*(x,y)+v_t^*(x)+y_t^*(y))$, существуют L_0, L_1, L_2 такие, что $\left|x_t^*(x,y)\right|\le L_0\|x\|\|y\|$, $\left|z_t^*(x)\right|\le L_1\|x\|$, $\left|y_t^*(y)\right|\le L_2\|y\|$ при $(x,y)\in X\times Y$ и $(x^*,v^*,y^*)\in D\varphi(\bar{x},\bar{y})$, то $\sup\limits_{t\in T}x_t^*(x,y)\ge x^*(x,y)$, $\sup\limits_{t\in T}(x_t^*(x,y)+v_t^*(x))\ge x^*(x,y)+v^*(x)$ и $\sup\limits_{t\in T}(x_t^*(x,y)+y_t^*(y))\ge x^*(x,y)+y^*(y)$ при $(x,y)\in X\times Y$.

Пусть $f: X\times Y\to R_{+\infty}$ 2-сублинейная функция. Положим

$$f^*(x^*,v^*,y^*)=\sup_{(x,y)\in X\times Y}\{x^*(x,y)+v^*(x)+y^*(y)-f(x,y)\}$$

при $(x^*,v^*,y^*)\in B(X\times Y,R)\times(X\times Y)^*$,

$$f^{**}(x,y)=\sup_{(x^*,v^*,y^*)\in B(X\times Y,R)\times(X\times Y)^*}\{x^*(x,y)+v^*(x)+y^*(y)-f^*(x^*,v^*,y^*)\}=$$

$$=\sup_{(x^*,v^*,y^*)\in domf^*}\{x^*(x,y)+v^*(x)+y^*(y)-f^*(x^*,v^*,y^*)\},$$

$$Df=\{(x^*,v^*,y^*)\in B(X\times Y,R)\times(X\times Y)^*:\ f(x,y)\ge x^*(x,y)+v^*(x)+y^*(y)\ \text{при}\ (x,y)\in X\times Y\}.$$

Так как $f^*(x^*,v^*,y^*)\ge x^*(x,y)+v^*(x)+y^*(y)-f(x,y)$ при $(x,y)\in X\times Y$, то получим, что $f(x,y)\ge f^{**}(x,y)$ при $(x,y)\in X\times Y$. Поэтому $Df^{**}\subset Df$.

Лемма 4.4. Если $F: X\otimes Y\times X\times Y\to R_{+\infty}$ сублинейная функция, $f(x,y)=F(x\otimes y,x,y)$ и $Df\ne\varnothing$, то

$$f^*(x^*,v^*,y^*)=\begin{cases}0; & (x^*,v^*,y^*)\in Df,\\ +\infty; & (x^*,v^*,y^*)\notin Df,\end{cases}\quad\text{и}\quad f^{**}(x,y)=\sup_{(x^*,v^*,y^*)\in Df}\{x^*(x,y)+v^*(x)+y^*(y)\}.$$

Следствие 4.1. Если $f: X\times Y\to R_{+\infty}$ собственная каноническая 2-сублинейная функция, то $f(x,y)=\sup\limits_{(x^*,v^*,y^*)\in Df}\{x^*(x,y)+v^*(x)+y^*(y)\}$ при $(x,y)\in X\times Y$.

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MEDICAL SCIENCES

COMPARATIVE ANALYSIS OF IMMUNOLOGICAL INDICATORS IN EARLY PREGNANT WOMEN WITH AND WITHOUT GENITAL INFECTIONS

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Abstract
This study examined changes in immunological parameters in women with and without genital infections in early pregnancy, up to 12 weeks. It was concluded that women with genital infections, compared to women without genital infections, exhibit less pronounced changes in cellular immunity in early pregnancy, up to 12 weeks, compared to proinflammatory and anti-inflammatory interleukins. These changes may indicate a lack of a significant proinflammatory immune response and may contribute to a favorable course of early pregnancy and the development of a full-term pregnancy.

Keywords: early pregnancy, genital infections, cellular immunity.

Adequate inflammation is essential for successful pregnancy outcome. In humans, proinflammatory responses, including the secretion of IL-6, IL-8, and TNF-α, are required for the acquisition of uterine receptivity [7]. Furthermore, elevated levels of IL-12, IL-1β, TNF-α, IL-6, and NO promote embryo attachment to the decidua [9].

Dendritic cells (DCs) are recruited to the uterus prior to implantation and modulate the cytokine profile at the fetal-maternal interface [5], and adequate DC-induced inflammation is necessary to ensure successful implantation and prevent first-trimester uterine miscarriage [1]. Overall, the uterine septum is considered a major risk factor for recurrent pregnancy loss, and low DC accumulation in the uterine septum may create a hostile immune environment in early pregnancy [8].

Macrophages with flexible plasticity also play an important role in early pregnancy [9]. They play an important role in the establishment of folliculogenesis. During the implantation period, macrophages produce proinflammatory cytokines. At the end of pregnancy, macrophages secreting proinflammatory cytokines are again required for parturition [9, 3]. Macrophages also play a crucial role in parturition, as they accumulate in the decidua before parturition in humans and rats [3]. Another study showed that macrophages are recruited to the human cervix and promote cervical ripening for vaginal delivery [6]. Thus, various inflammatory and immune responses are expected to occur during pregnancy and for successful delivery.

Sterile inflammation and innate immune cells also play an important role in the occurrence of miscarriage and infertility. In particular, these complications are believed to be related to DC function. In humans, DC are composed of monocyte-derived plasmacytoid DC and conventional (classical) DC subtypes [4]. These results indicate that activated innate immune cells (DCs) and natural killer (iNKT) cells directly lead to miscarriage without microbial-associated inflammation. Thus, innate immune cells, DCs, and iNKT cells are associated not only with preterm birth but also with miscarriage in the absence of infection.

The impact of infections in early pregnancy remains controversial, as some studies indicate an increased risk of miscarriage, while others show no increased risk. Therefore, further research is needed to determine whether specific infections actually increase the risk of miscarriage. It has been suggested that not simply the presence of bacteria themselves, but differences in the host response to genital infections, may contribute to the increased risk of preterm birth [2].

Objective: To examine changes in immunological parameters in women with and without genital infections in early pregnancy.

Materials and Methods: A total of 3760 women were examined, divided into two groups. Group 1 included 32 women who had had a full-term pregnancy and delivered a full-term birth, and who had no genital infections before pregnancy. Group 2 included 28 women who had had a full-term pregnancy and delivered a full-term birth but who had genital infections (Chlamydia trachomatis, Ureaplasma urealyticum, Mycoplasma hominis) before pregnancy.

Blood tests were performed using ELISA before pregnancy, after 6 weeks, and after 12 weeks: pro-inflammatory (interleukin-1β (IL-1) and tumor necrosis factor-α (TNF-α)), and anti-inflammatory (interleukin-10 (IL-10)). In addition, the immunogram was examined, and indicators of chlamydia (DNA Chlamydia trachomatis), mycoplasmosis (DNA Mycoplasma hominis) and ureaplasmosis (DNA Ureaplasma urealyticum) were determined in a scraping from the cervix before pregnancy using the PCR method.

Results and discussion. The obtained results of the study of proinflammatory and anti-inflammatory interleukins revealed that in women without genital infections, proinflammatory interleukin TNF-α levels were significantly higher at 6 weeks of pregnancy than at 6 weeks of pregnancy. However, in women without

genital infections, these levels at 12 weeks of pregnancy were not significantly higher than at 6 weeks of pregnancy and were significantly higher than at 6 weeks of pregnancy.

When studying TNF-α levels in women with genital infections at 6 weeks of pregnancy, these levels were significantly higher than at 6 weeks of pregnancy and were significantly higher than at 6 weeks of pregnancy. Furthermore, in women with genital infections, TNF-α levels before pregnancy were significantly higher than at 6 weeks of pregnancy and were significantly higher than at 6 weeks of pregnancy. Moreover, in women without infections, TNF-α levels before pregnancy were significantly higher than at 6 weeks of pregnancy (Table 1).

A similar pattern of changes in IL-1β levels was observed in women with and without genital infections (Table 1).

At the same time, in the blood of women without genital infections, levels of the anti-inflammatory interleukin IL-10 at 6 weeks of pregnancy were not significantly lower than similar levels before pregnancy.

Table 1.

Changes in immunological parameters in women in early pregnancy

INTERLEUKINS				
Study parameters	**Group**	**Pre-pregnancy**	**6 weeks of pregnancy**	**12 weeks of pregnancy**
ФНО-α пг/мл	1	6,2±0,8	9,7±1,2*	11,9±1,5 *
	2	9,6±1,1°	12,7±1,4*	15,2±1,6 *
ИЛ-1β пг/мл	1	3,7±0,4	7,2 ± 0,9*	10,1±1,2*
	2	6,5±0,8 °	10,3 ± 1,2*	13,2±1,4*
ИЛ-10 пг/мл	1	9,3±1,2	7,4±0,8	5,9±0,7*
	2	8,5±1,0	6,3±0,7	4,6±0,5*

IMMUNOGRAM						
Indicators	**Normal**		**Group**	**Pre-pregnancy**	**6 weeks pregnant**	**12 weeks pregnant**
Leukocytes	В 1 мкл	4000-10000	1	5200±463	4700±418	5450±457
			2	5600±519	5950±536	6200±558
Lymphocytes	%	20-40	1	35±3,1	31±2,6	28±2,4
			2	31±2,5	23±1,8	19±1,6*
	В 1 мкл	800-4000	1	1820±159	1457±127	1526±139
			2	1736±148	1369±113	1178±94*
B lymphocytes (CD20)	%	10-36	1	32±2,9	27±2,4	25±2,3
			2	27±2,3	21±1,6*	17±1,4*
	В 1 мкл	162-700	1	582±53	393±34*	381±33*
			2	468±38	287±23*	200±15*
T lymphocytes (CD3)	%	48-80	1	56±5,1	49±4,3	44±3,7
			2	48±4,5	43±3,9	39±3,5
	В 1 мкл	800-2400	1	1019±95	713±64*	623±52*
			2	833±74	588±55*	459±38*
T helper cells (CD4)	%	24-42	1	25±2,1	23±1,7	20±1,5
			2	22±1,6	21±1,4	19±1,3
T suppressor cells (CD8)	%	14-29	1	15±1,2	17±1,3	18±1,6
			2	16±1,4	18±1,5	20±1,7
CD4/CD8		1,2-2,0	1	1,7±0,13	1,4±0,11	1,1±0,09*
			2	1,4±0,11	1,2± 0,08	1,0±0,07*
Natural killer cells - CD16	%	4-27	1	9,0±0,6	11±0,9	13±1,1*
			2	10,0±0,7	12,0±0,8	14,0±1,2*
CD25 activated lymphocytes	%	7-18	1	11±0,8	13±1,0	15±1,3*
			2	12±0,9	14±1,1	17±1,4*
CD95 (apoptosis marker)	%	34-46	1	15±1,2	12±0,9	10±0,7*
			2	13±1,1	10±0,7*	9±0,6*

Note: 1 - women with a full-term pregnancy without genital infections; 2 - women with a full-term pregnancy with genital infections.

* - significantly different values from pre-pregnancy values.

o - significantly different values from Group 1 values.

Moreover, this indicator in women at 12 weeks of pregnancy was not significantly lower than at 6 weeks of pregnancy and was significantly lower than similar results before pregnancy. IL-10 levels in women with genital infections after 6 weeks of pregnancy were also not significantly lower than similar results before pregnancy. These results at 12 weeks of pregnancy were also not significantly lower than at 6 weeks of pregnancy and were significantly lower than similar results before pregnancy. Moreover, IL-10 levels in women with genital infections before pregnancy were not significantly lower than similar results without infections (Table 1).

A study of cellular immunity (Table 1) revealed no significant differences in the white blood cell counts in women without genital infections and women with genital infections. There were also no significant differences in both groups between women at 6 and 12 weeks of pregnancy compared to women before pregnancy. When examining the relative lymphocyte count (%), a non-significant decrease was observed in women without genital infections at 6 weeks of pregnancy and a greater decrease at 12 weeks compared to pre-pregnancy levels. A similar, but more pronounced, decrease in lymphocyte count (%) was observed in women with genital infections. However, there was a significant difference in women at 12 weeks of pregnancy compared to women before pregnancy. When examining the absolute lymphocyte count (per μL), a similar non-significant decrease was observed in women without genital infections and a significant decrease at 12 weeks of pregnancy compared to women before pregnancy (Table 1).

When examining B-lymphocytes (CD20) (%) (Table 1), a non-significant decrease was also observed in women without genital infections at 6 weeks of pregnancy and a greater decrease at 12 weeks of pregnancy compared to pre-pregnancy levels. At the same time, women with genital infections showed a significant trend toward a significant decrease in B-lymphocyte counts (%) at both 6 and 12 weeks of pregnancy compared to pre-pregnancy levels. However, absolute B-lymphocyte counts (per μL) showed a significant decrease at 6 and 12 weeks of pregnancy, both in women without genital infections and in women with genital infections, compared to women before pregnancy. When examining T-lymphocyte counts (CD3) in %, a non-significant decrease was observed at 6 and 12 weeks of pregnancy in both women without genital infections and in women with genital infections, compared to women before pregnancy. At the same time, a reliable decrease in T-lymphocytes in 1 μl values was noted at 6 and 12 weeks of pregnancy, both in women without genital infections and in women with genital infections in relation to women before pregnancy (Table 1).

The obtained results of the study of T-helpers (CD4) in relative indicators (Table 1) revealed a non-significant trend of decrease at 6 and 12 weeks of pregnancy, both in women without genital infections, and in women with genital infections, in relation to women before pregnancy. At the same time, both in women without genital infections, and in women with genital infections, in relation to women before pregnancy, a non-significant trend of increase in T-suppressors (CD8) at 6 and 12 weeks of pregnancy was noted. According to the study of the CD4/CD8 ratio, a decrease in this indicator was noted at 6 and 12 weeks of pregnancy, both in women without genital infections and in women with genital infections in relation to women before pregnancy. With a significant decrease at 12 weeks of pregnancy and a lower level of indicators in women with a full-term pregnancy with genital infections (Table 1). At the same time, an increase in natural killer (CD16) cells was detected in women with full-term pregnancy without genital infections and in women with genital infections compared to women before pregnancy. This was accompanied by a significant increase at 12 weeks of pregnancy and a higher level of indicators in women with genital infections. A similar trend in changes in CD16 was noted in the study of activated lymphocytes (CD25). However, when studying the apoptosis marker CD95, a decrease in this indicator was noted at 6 and 12 weeks of pregnancy, both in women without genital infections and in women with genital infections compared to women before pregnancy. This was accompanied by a significant decrease at 12 weeks of pregnancy in women without genital infections and a significant decrease at 6 and 12 weeks of pregnancy in women with genital infections, as well as a lower level of indicators in women with genital infections (Table 1).

The data obtained show that blood levels of proinflammatory interleukins TNF-α and IL-1β in both women without genital infections and those with genital infections increased significantly after 6 weeks of pregnancy and even more significantly after 12 weeks of pregnancy compared to similar values before pregnancy. However, this indicator in women with genital infections was not significantly higher than in women without genital infections, and was significantly higher before pregnancy. At the same time, blood levels of anti-inflammatory interleukin IL-10 showed the opposite trend, decreasing non-significantly at 6 weeks and significantly at 12 weeks of pregnancy in both women without genital infections. These indicators were also non-significantly lower in women with genital infections than in women without genital infections. Cellular immunity studies revealed insignificant changes in leukocyte counts in both women without genital infections and those with genital infections. At the same time, a significant increase in lymphocytes was noted at 12 weeks of pregnancy in women with genital infection compared to women before pregnancy, both in relative terms in % and in absolute terms in 1 μl, which had an inverse direction of changes in relation to pro-inflammatory interleukins TNF-α and IL-1β and the same direction in relation to anti-inflammatory interleukins IL-10. A similar direction of changes was observed in the study of both B-lymphocytes and T-lymphocytes with an insignificant decrease compared to the pre-pregnancy results at 6 and 12 weeks of pregnancy in B-lymphocytes and T-lymphocytes in % in women without genital infections, as well as T-lymphocytes in % in

women with genital infections. A significant decrease in absolute B-lymphocyte and T-lymphocyte counts per 1 μL was also observed at 6 and 12 weeks of pregnancy in women with and without genital infections. These changes in B-lymphocyte and T-lymphocyte counts also had an inverse direction of modifications in relation to the pro-inflammatory interleukins TNF-α and IL-1β and the same direction in relation to the anti-inflammatory interleukins IL-10. However, no significant change was observed with a decrease in T-helper cells at 6 and 12 weeks of pregnancy in women with and without genital infections. No significant change was observed with an increase in T-suppressor cells at 6 and 12 weeks of pregnancy in women with and without genital infections. At the same time, a non-significant decrease in the CD4/CD8 ratio was observed in women both without and with genital infections at 6 weeks of pregnancy and a significant decrease at 12 weeks of pregnancy. However, natural killer and activated lymphocyte counts increased non-significantly at 6 weeks of pregnancy and significantly at 12 weeks of pregnancy in women both without and with genital infections. Changes in the apoptosis marker were manifested by a non-significant decrease at 6 weeks of pregnancy and a significant decrease at 12 weeks of pregnancy in women without genital infections, as well as a significant decrease at 6 and 12 weeks of pregnancy in women with genital infections. Thus, women with genital infections experienced less pronounced changes in cellular immunity compared to women with genital infections, compared to women with genital infections.

Conclusions. Women with genital infections, compared to women without genital infections, show less pronounced changes in cellular immunity in the early stages up to 12 weeks of pregnancy, compared to proinflammatory and anti-inflammatory interleukins. These changes may indicate a lack of a significant pro-inflammatory immune response and may contribute to a favorable course of early pregnancy and the development of a full-term pregnancy.

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PHILOLOGICAL SCIENCES

GEORGIA EPITAPHS OF THE JEWS OF KUTAISI

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Abstract

Tombstones are important monuments that refect the features of national material and spiritual culture. "Stone inscriptions" not only indicate the identity, gender, dates of birth and death of individuals, but also inform us about the social status, surnames, anthroponyms, toponymy, vocabulary, phraseology, and dialectisms widespread in a given village or community; they also show the human relationship with the earthly life, and indicate their profession, and religious as well as ideological infuences. Stones of various forms and sizes represent the source of calligraphy, stone carving, ornaments, cultural infuences, and blending. The proposed paper deals with the analysis of types and forms of epitaphs refected on tombstones (dated back to the 19th–20th cc.), discovered by us during expeditions in diferent parts of Georgia, specifcally:

1. narrative epitaphs;
2. rhyme-type epitaphs: a) rhymes authored by mourners; b) celebrated verses of poets and quotations of famous individuals;
3. Paraphrased prayers. of writing the stories of life on tombstones, to a greater or lesser extent, is still alive. However, it may disappear. In such a case, monotony will prevail. Tombstones, similar to the epigraphic monuments, represent cultural legacy afected by threats in the ghreats in the globalization era. Therefore, it is crucial to ensure their description and examination.

Epitaphs at the Sapichkhia Jewish cemetery are inscribed in Hebrew, Russian and Georgian – either in two languages smultaneously (Hebrew and Georgian or Georgian and Russian) or in Georgian and Hebrew alone.In this article we will not discuss Russian-language epitaphs.

Keywords: Georgian Jews; Kutaisi; Jewish cemetery; Sapichkhia Cemetery; epitaphs; epigraphy; tombstone inscriptions; funerary culture; cultural heritage; Hebrew inscriptions; Georgian inscriptions; multilingual epitaphs; Jewish history; onomastics; memorial traditions.

It is widely known that the study of epigraphy yields interesting information for various fields – philology, history, history of religion, ethnography and onomastics. Epigraphic monuments, as tangible remnants of the past, reflect the early cultural-historical situation of a given country without any alteration. Epigraphy emerged as a scholarly discipline in the XIXth century, when researchers began publishing corpora of Greek and Latin inscriptions, as well as deciphering and publishing inscriptions discovered in Eastern countries. Among these researcwehrs were Dubois de Montpereux, M. Brosset, N.Marr, D. Bakradze, E. Takaishvili, P. Ioseliani, A. Khakhanashvili, and others. The study of epigraphic monuments is of particular importance for clarifying certain aspects of Georgian history. At times, they constitute the sole written source available. They are characterized by great precision in recording facts and enable us to establish a range of interesting historical details. They represent authentic primary written sources from the past that have remained free from the alterations of copyists.

The reality characteristic of that era of cultural life will vanish. Tombstones, as epigraphic monuments, constitute cultural heritage at risk in the age of globalization; therefore, their documantation and study is a matter of urgency.

In the 1860s, the current Sapichkhia cemetery did not exist; the area was a stretch of forest visited primarily by people gathering brushwood. The city's expansion led to the development of the site. A century and a half ago, "international" cemeteries – serving Georgians, Catholics, Armenians, Jews, and others – were established in a section of the forest that had remained uncleared. In 1894, the city authorities of the time opened a new cemetery on a mountain plateau overlooking the Tskhatsitela River; it remains in use to this day (newspaper "Industrial Kutaisi", November 16, 1967; author: Sh.Lomtadze).

A large section of the Sapichkhia Cemetery in Kutaisi – a historic burial ground for Georgian Orthodox Christians and Catholics – is occupied by the graves of Jewish community; this ares remains unstudied and is therefore of great interest for exploring the centuries-old tradition of Georgian-Jewish relations. Among those buried at the Sapichkhia Cemetery are Jewish writers, doctors, teachers, and ordinary people, who, together with Georgians shaped the history of Kutaisi.

The tombstone is a monument of widespread memorial culture. It is an important object reflecting the peculiarities of the material and spiritual culture of a nation. Regarding epitaphs, we have learned that stone inscriptions mark not only the identity of people, their gender, and the time of birth and death, instead, they tell us about social status, common surnames, the changing of a woman's surname upon marriage, anthroponyms, toponymy, vocabulary and phraseology,

and dialectisms, as well as about human sorrow, attitudes toward the fleeting nature of life, professions, and religious and ideological influences. Stones of various shapes and sizes serve as promary sources of information regarding calligraphy and stone carving, ornamentation, symbolism, the blending of cultural influences, and economic conditions.

Epitaphs at the Sapichkhia Jewish cemetery are inscribed in Hebrew, Russian and Georgian – either in two languages smultaneously (Hebrew and Georgian or Georgian and Russian) or in Georgian and Hebrew alone.In this article we will not discuss Russian-language epitaphs.

In this article, we examine the forms of epitaphs found on tombstones in Kutaisi – specifically at the Sapichkhia Cemetery – which can be categorized as follows:

1. Narrative epitaphs;
2. Poetic epitaphs, comprising two subgroups:a)poems written by the bereaved, b) well-known poems by poets c) quotations from the works of prominent figures;
3. Paraphrased prayers serving as epitaphs.

Sapichkhia Jewish Graves:

a) The upper part of the stone bears an inscription in Hebrew, with a corresponding Georgian inscription alongside it that reads as follows:

"Here rests Gabriel
[Second line] – son of Ariamin, Rizhinashvili
[Third line] – aged 34, died in 1920"

These last 3 words are positioned lower down, towards the base of the tombstone. The tombstone is a horizontal, white, dressed stone measuring 2 meters in length and 1.20 meters in height, It is divided into 3 vertical tiers: the middle tier is reserved for the inscription, while the upper tier features carvings on the sides – all executed in bas-relief. As for the inscription itself, it is incised; the deeply cut letters follow Gogebashvili's four-line calligraphic style, with the exception of the letter "o", which is rendered with angular contours.

b) The inscription on this grave is solely in Georgian: "Here restd Sara QGH..u (Letters are erased) (Picture #1)

Picture N1

[Second line] – daughter of Khis Qiuri, wife
[Third line] – of Gabriel Eligulashvili
[Forth line] – born August 30, 1912, aged 34 at the time of death; died in 1920"

These 3 words are deeply incised; the stone is surrounded by 4 bas-reliefs of grape clusters.. The tombstone is a horizontal, white, dressed stone measuring 2 meters in length and 1.80 meters in height, It is divided into 3 vertical tiers: the middle tier is reserved for the inscription; the inscription itself is incised, featuring deeply cut letters exe uted in Gogebashvili's four-line calligraphic style.

At the Sapichkhia cemetery, there are numerous tombstones featuring bas-reliefs of grapevines; this indicates that Jews embraced Georgian culture quite early on and, like the Georgians, pressed grapes and produced wine – though their wine differed from thr Georgian variety and was known as "Kosher" wine.

One encounters graves that bear only the deceased's full name, dates of birth and death, and the name of the person who erected the stone.

There are graves bearing only the deceased's first and last names, dates of birth and death, and the name of the person who erected the stone:

a)There is a Hebrew inscription on the left side, and a Georgian inscription on the right side:

First line – Tina, Daughter of Shineli
Second line – Shaloma
Third line – Rijinashvili; 11 November, 1979 (Picture #2)

Picture N2

b) First line – Tiko,, Daughter of Beniamin
Second line – Tavidashvili
Third line – Pichkhadze
Fourth line – 1910-1988 (Picture #3)

Picture N3

2.Verse-style epitaphs

2.1. Verses written by the bereaved

On encounters epitaphs that might be termed "verses" only in a loose sense, as the majority are – to all intents and purposes – "pseudo-poems", entirely devoid of rhyme,ehythm, or artistic imagery. Such epitapha were typically composed by a local figure known as a "melekse" (verse-maker).

First line - My young wife and children
Second line – That they made me weep
Third line – my poor father
Fourth line – my elderly parent
Fifth line – with death
Sixth line – comes weeping to my
Seventh line - grave
Eight line – mourns over me and towards me
Ninth line – bears a soul and a heart (Picture #4)

Picture N4

The epitaph is inscribed in the modern "Mkhedruli" script, with the letters standing individually and unlinked.It follows a two-line layout, with all letters of uniform size. Leafy brenches adorn the top and sides of the marble slab, while an oval image is featured in the center.

The epitaphs we have documented are significant not only froma linguistic perspective but also for the study of mentality and worldview; they reflect the psychology of the ethnic group as a whole, rather than merely that of individuals. It can be said that the tradition of inscribing a person's life story – their journey through this world – onto gravestones persists to some extent today, yet there is a risk that this practice could gradually be lost.

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